Example 6.1—Flow Between Parallel Plates

Fig. E6.1.1 shows a fluid of viscosity μ that flows in the x direction between two rectangular plates, whose width is very large in the z direction when compared to their separation in the y direction. Such a situation could occur in a die when a polymer is being extruded at the exit into a sheet, which is subsequently cooled and solidified. Determine the relationship between the flow rate and the pressure drop between the inlet and exit, together with several other quantities of interest.

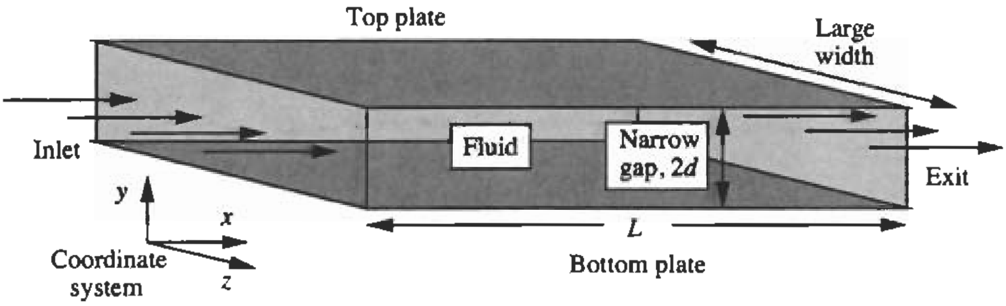


Fig. E6.1.1 Geometry for flow through a rectangular duct. The spacing between the plates is exaggerated in relation to their length.

Assumptions:

1. As already stated, it is steady and Newtonian, with constant density and viscosity. (These assumptions will often be taken for granted, and not restated, in later problems.)
2. There is only one nonzero velocity component—that in the direction of flow, v_x. Thus, $v_y = v_z = 0$.
3. Since, in comparison with their spacing, $2d$, the plates extend for a very long distance in the z direction, all locations in this direction appear essentially identical to one another. In particular, there is no variation of the velocity in the z direction, so that $\partial v_x/\partial z = 0$.
4. Gravity acts vertically downwards; hence, $g_y = -g$ and $g_x = g_z = 0$.
5. The velocity is zero in contact with the plates, so that $v_x = 0$ at $y = \pm d$.

Solution:

$v_y = v_z = 0$

Continuity Equation

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{ss} + \frac{\partial(\rho v_x)}{\partial x} + \overbrace{\cancel{\frac{\partial(\rho v_y)}{\partial y}}}^{0} + \overbrace{\cancel{\frac{\partial(\rho v_z)}{\partial z}}}^{0} = 0,$$

$$\Rightarrow \quad \frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow \quad v_x \neq f(x)$$

also, for wide plates in z-direction,

$$v_x \neq f(z)$$

Navier-Stokes Equations:

ss $\quad \frac{dV_x}{dx} = 0 \quad\quad V_x \neq f(x)$

$$\rho\left(\cancelto{0}{\frac{\partial v_x}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_x}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_x}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancelto{0}{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_x}{\partial z^2}}\right) + \cancelto{0}{\rho g_x},$$

$$\rho\left(\cancelto{0}{\frac{\partial v_y}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_y}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_y}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancelto{0}{\frac{\partial^2 v_y}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_y}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_y}{\partial z^2}}\right) + \rho g_y,$$

$$\rho\left(\cancelto{0}{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_z}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_z}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left(\cancelto{0}{\frac{\partial^2 v_z}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right) + \cancelto{0}{\rho g_z}.$$

simplify:

$$\frac{d^2 V_x}{dy^2} = \frac{1}{\mu}\frac{\partial P}{\partial x} \quad (1)$$

$$\frac{\partial P}{\partial y} = -\rho g \quad (2)$$

$$\frac{\partial P}{\partial z} = 0 \quad (3) \Rightarrow P \neq f(z)$$

Pressure Profile:

const. of integration

integrate (2) $\Rightarrow$ $P = -\rho g y + f(x)$

In pressure driven flows like this problem, the pressure changes linearly along the direction of the flow:

$$P = -\rho g y + (a + b x)$$

constants

$x = 0 \qquad P = P_1 \Rightarrow P_1 = -\rho g y + (a)$

$x = L \qquad P = P_2 \Rightarrow P_2 = -\rho g y + (a + bL)$

$$\Downarrow$$

$$a = P_1$$

$$b = \frac{P_2 - P_1}{L} = \frac{\Delta P}{L}$$

$$\Rightarrow \boxed{P = -\rho g y + \frac{\Delta P}{L} x + P_1} \qquad (4)$$

Velocity Profile:

$$\text{Eq } (1) \Rightarrow \frac{d^2 V_x}{dy^2} = \frac{1}{\mu} \frac{\partial P}{\partial x}$$

$$\text{From } (4) \quad \frac{\partial P}{\partial x} = \frac{\Delta P}{L}$$

$$\Rightarrow \frac{d^2 V_x}{dy^2} = \underbrace{\frac{1}{\mu} \frac{\Delta P}{L}}_{\text{constant}}$$

integrate once:

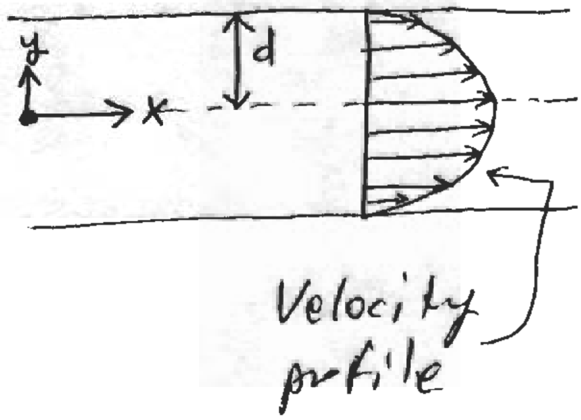
integration constant

$$\Rightarrow \frac{dV_x}{dy} = \frac{1}{\mu} \frac{\Delta P}{L} y + C_1$$

integrate again:

$$\Rightarrow V_x = \frac{1}{2} \frac{\Delta P}{\mu L} y^2 + C_1 y + C_2$$

boundary conditions:



$y = 0$ $\qquad \dfrac{dV_x}{dy} = 0 \qquad$ (BC1)

Velocity is Maximum at $y=0$.

$y = d$ $\qquad V_x = 0 \qquad$ (BC2)

no slip condition

recall,

$$\frac{dV_x}{dy} = \frac{1}{\mu}\frac{\Delta P}{L}\, y + C_1 \qquad \text{apply BC1}$$

$$\Rightarrow 0 = \frac{1}{\mu}\frac{\Delta P}{L}(0) + C_1 \Rightarrow C_1 = 0$$

recall,

$$V_x = \frac{1}{2}\frac{\Delta P}{\mu L}\, y^2 + \cancel{C_1 y} + C_2 \qquad \text{apply BC2}$$

$$0 = \frac{1}{2}\frac{\Delta P}{\mu L}\, d^2 + C_2$$

$$\Rightarrow C_2 = -\frac{1}{2\mu}\frac{\Delta P}{L}\, d^2$$

$$\Rightarrow \boxed{V_x = \frac{1}{2\mu}\frac{\Delta P}{L}\left(y^2 - d^2\right)}$$

Parabolic Velocity Profile

Volumetric Flow Rate:

$$Q = \int_A v_x \, dA$$

A ↑ area integral

dA ↑ area normal to direction of flow

$$\Rightarrow \quad Q = \int_0^W \int_{-d}^{+d} v_x \, dy \, dz = W \int_{-d}^{d} v_x \, dy.$$

$$Q = \int_{-d}^{d} v_x \, dy$$

(Volumetric flow rate per unit width $W=1$)

$$= \frac{1}{2\mu} \frac{\Delta P}{L} \int_{-d}^{+d} (y^2 - d^2) \, dy$$

$$\boxed{Q = -\frac{2}{3} \frac{\Delta P}{\mu L} d^3}$$

(do it yourself)

Maximum Velocity:

$$V_{max} = V(y=0) = -\frac{1}{2\mu} \frac{\Delta P}{L} d^2$$

Mean Velocity (Average):

$$V_m = \frac{Q}{A} = -\frac{1}{3} \frac{\Delta P}{\mu L} d^2$$

$A = 2d * 1$

$$\frac{V_m}{V_{max}} = \frac{\frac{1}{3}\frac{1}{\mu}\frac{\Delta P}{L} d^2}{\frac{1}{2}\frac{1}{\mu}\frac{\Delta P}{L} d^2} = \frac{2}{3}$$

$$\Rightarrow \boxed{V_m = \frac{2}{3} V_{max}}$$

Shear Stress:

$$\tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \cancelto{0}{\frac{\partial v_y}{\partial x}} \right)$$

$$\boxed{\tau_{yx} = \frac{\Delta P}{L} y}$$ linear

$\Delta P = -$ ve

$\Rightarrow$ if y is positive τ_{yx} is $-$ve

" " " negative τ_{yx} is $+$ve