Example 2:

$$\frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+11\frac{dy}{dt}+6y=4\frac{du}{dt}+2u$$

$$y(0)=y'(0)=y''(0)=0$$

$$\frac{du}{dt}=0 \quad at \quad t=0$$

system at rest (s.s.)

To find transient response for u(t) = unit step at t > 0

1. Take Laplace Transform (L.T.)
2. Factor, use partial fraction decomposition
3. Take inverse L.T.

Step 1 Take L.T. (note zero initial conditions)

$$s^3Y(s)+6s^2Y(s)+11sY(s)+6Y(s)=4sU(s)+2U(s)$$

Partial Fraction Expansions

Basic idea: Expand a complex expression for $Y(s)$ into simpler terms, each of which appears in the Laplace Transform table. Then you can take the L^{-1} of both sides of the equation to obtain $y(t)$.

Example:

$$Y(s) = \frac{s+5}{(s+1)(s+4)} \tag{3-41}$$

Perform a partial fraction expansion (PFE)

$$\frac{s+5}{(s+1)(s+4)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} \tag{3-42}$$

where coefficients α_1 and α_2 have to be determined.

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To find α_1 *: Multiply both sides by s + 1 and let s = -1*

$$\therefore \quad \alpha_1 = \left.\frac{s+5}{s+4}\right|_{s=-1} = \frac{4}{3}$$

To find α_2 *: Multiply both sides by s + 4 and let s = -4*

$$\therefore \quad \alpha_2 = \left.\frac{s+5}{s+1}\right|_{s=-4} = -\frac{1}{3}$$

A General PFE

Consider a general expression,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\overset{n}{\underset{i=1}{\pi}}(s+b_i)} \qquad \text{(3-46a)}$$

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Here $D(s)$ is an n-th order polynomial with the roots $(s = -b_i)$ all being *real* numbers which are *distinct* so there are no repeated roots.

The PFE is:

$$Y(s) = \frac{N(s)}{\prod_{i=1}^{n}(s+b_i)} = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i} \qquad \text{(3-46b)}$$

Note: $D(s)$ is called the "characteristic polynomial".

Special Situations:

Two other types of situations commonly occur when $D(s)$ has:

i) Complex roots: e.g., $b_i = 3 \pm 4j \qquad \left(j \triangleq \sqrt{-1}\right)$

ii) Repeated roots (e.g., $b_1 = b_2 = -3$)

For these situations, the PFE has a different form. See SEM text (pp. 61-64) for details.

Example 3.2 (continued)

Recall that the ODE, $\dddot{y} + +6\ddot{y} + 11\dot{y} + 6y = 1$, with zero initial conditions resulted in the expression

$$Y(s) = \frac{1}{s\left(s^3 + 6s^2 + 11s + 6\right)} \qquad \text{(3-40)}$$

The denominator can be factored as

$$s\left(s^3 + 6s^2 + 11s + 6\right) = s(s+1)(s+2)(s+3) \qquad \text{(3-50)}$$

Note: Normally, numerical techniques are required in order to calculate the roots.

The PFE for (3-40) is

$$Y(s) = \frac{1}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3} \qquad \text{(3-51)}$$

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Step 2b. Use partial fraction decomposition

$$\frac{4s+2}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}$$

Multiply by s, set s = 0

$$\left.\frac{4s+2}{(s+1)(s+2)(s+3)}\right|_{s=0} = \alpha_1 + s\left.\left[\frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}\right]\right|_{s=0}$$

$$\frac{2}{1\cdot 2\cdot 3} = \alpha_1 = \frac{1}{3}$$

For α_2, multiply by (s+1), set s=-1 (same procedure for α_3, α_4)

$$\alpha_2 = 1, \quad \alpha_3 = -3, \quad \alpha_4 = \frac{5}{3}$$

Step 3. Take inverse of L.T. $\left(Y(s) = \frac{1}{3s} + \frac{1}{s+1} - \frac{3}{s+2} + \frac{5/3}{s+3}\right)$

Take L^{-1} of both sides:

$$L^{-1}\left[Y(s)\right] = L^{-1}\left[\frac{1/6}{s}\right] - L^{-1}\left[\frac{1/2}{s+1}\right] + L^{-1}\left[\frac{1/2}{s+2}\right] + L^{-1}\left[\frac{1/6}{s+3}\right]$$

From Table 3.1,

$$y(t) = \frac{1}{3} + e^{-t} - 3e^{-2t} + \frac{5}{3}e^{-3t}$$

$$t \to \infty \qquad y(t) \to \frac{1}{3}$$

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You can use this method on any order of ODE, limited only by factoring of denominator polynomial (characteristic equation)

Must use **modified** procedure for repeated roots, imaginary roots

Important Properties of Laplace Transforms

1. Final Value Theorem

It can be used to find the steady-state value of a closed loop system (providing that a steady-state value exists.

Statement of FVT:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[sY(s)\right]$$

providing that the limit exists (is finite) for all $\mathrm{Re}(s) \geq 0$, where Re (s) denotes the real part of complex variable, s.

2. Initial Value Theorem

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Example:

Suppose,

$$Y(s) = \frac{5s+2}{s(5s+4)} \qquad (3\text{-}34)$$

Then,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\frac{5s+2}{5s+4}\right] = 0.5$$

3. *Time Delay*

Time delays occur due to fluid flow, time required to do an analysis (e.g., gas chromatograph). The delayed signal can be represented as

$$y(t-\theta) \qquad \theta = \text{time delay}$$

Also,

$$\mathcal{L}\left[y(t-\theta)\right] = e^{-\theta s} Y(s)$$