Chapter 3

Laplace Transforms

1. Standard notation in dynamics and control (shorthand notation)

2. Converts mathematics to algebraic operations

3. Advantageous for block diagram analysis

Laplace Transforms

- Important analytical method for solving *linear* ordinary differential equations.
    - Application to nonlinear ODEs? Must linearize first.
- Laplace transforms play a key role in important process control concepts and techniques.
    - Examples:
        - Transfer functions
        - Frequency response
        - Control system design
        - Stability analysis

Definition

The Laplace transform of a function, $f(t)$, is defined as

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^{\infty} f(t)e^{-st}dt \qquad (3\text{-}1)$$

where $F(s)$ is the symbol for the Laplace transform, $\mathcal{L}$ is the Laplace transform operator, and $f(t)$ is some function of time, t.

Note: The $\mathcal{L}$ operator transforms a time domain function $f(t)$ into an s domain function, $F(s)$. s is a *complex variable*:

$s = a + bj$,

$j \triangleq \sqrt{-1}$

Inverse Laplace Transform, $\mathcal{L}^{-1}$:

By definition, the inverse Laplace transform operator, $\mathcal{L}^{-1}$, converts an *s*-domain function back to the corresponding time domain function:

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right]$$

Important Properties:

Both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are *linear operators*. Thus,

$$\begin{aligned} \mathcal{L}\left[ax(t)+by(t)\right] &= a\mathcal{L}\left[x(t)\right]+b\mathcal{L}\left[y(t)\right] \\ &= aX(s)+bY(s) \qquad (3\text{-}3) \end{aligned}$$

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where:

- $x(t)$ and $y(t)$ are arbitrary functions
- a and b are constants
- $X(s) \triangleq \mathcal{L}\left[x(t)\right]$ and $Y(s) \triangleq \mathcal{L}\left[y(t)\right]$

Similarly,

$$\mathcal{L}^{-1}\left[aX(s)+bY(s)\right]=ax(t)+by(t)$$

Laplace Transforms of Common Functions

1. Constant Function

Let $f(t) = a$ (a constant). Then from the definition of the Laplace transform in (3-1),

$$\mathcal{L}(a) = \int_0^\infty a e^{-st} dt = -\frac{a}{s} e^{-st} \Big|_0^\infty = 0 - \left(-\frac{a}{s}\right) = \boxed{\frac{a}{s}} \qquad (3\text{-}4)$$

2. Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

$$S(t) \triangleq \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \qquad (3\text{-}5)$$

Because the step function is a special case of a “constant”, it follows from (3-4) that

$$\mathcal{L}\left[S(t)\right] = \frac{1}{s} \qquad (3\text{-}6)$$

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3. Derivatives

This is a very important transform because derivatives appear in the ODEs we wish to solve. In the text (p.53), it is shown that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \qquad (3\text{-}9)$$

initial condition at $t = 0$

Similarly, for higher order derivatives:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (3\text{-}14)$$

where:

- n is an arbitrary positive integer

- $f^{(k)}(0) \triangleq \left. \dfrac{d^k f}{dt^k} \right|_{t=0}$

Special Case: All Initial Conditions are Zero

Suppose $f(0) = f^{(1)}(0) = \ldots = f^{(n-1)}(0)$. Then

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s)$$

In process control problems, we usually assume zero initial conditions. *Reason:* This corresponds to the nominal steady state when "deviation variables" are used, as shown in Ch. 4.

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4. Exponential Functions

Consider $f(t) = e^{-bt}$ where $b > 0$. Then,

$$\mathcal{L}\left[e^{-bt}\right] = \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt$$

$$= \frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty = \boxed{\frac{1}{s+b}} \qquad (3\text{-}16)$$

5. Rectangular Pulse Function

It is defined by:

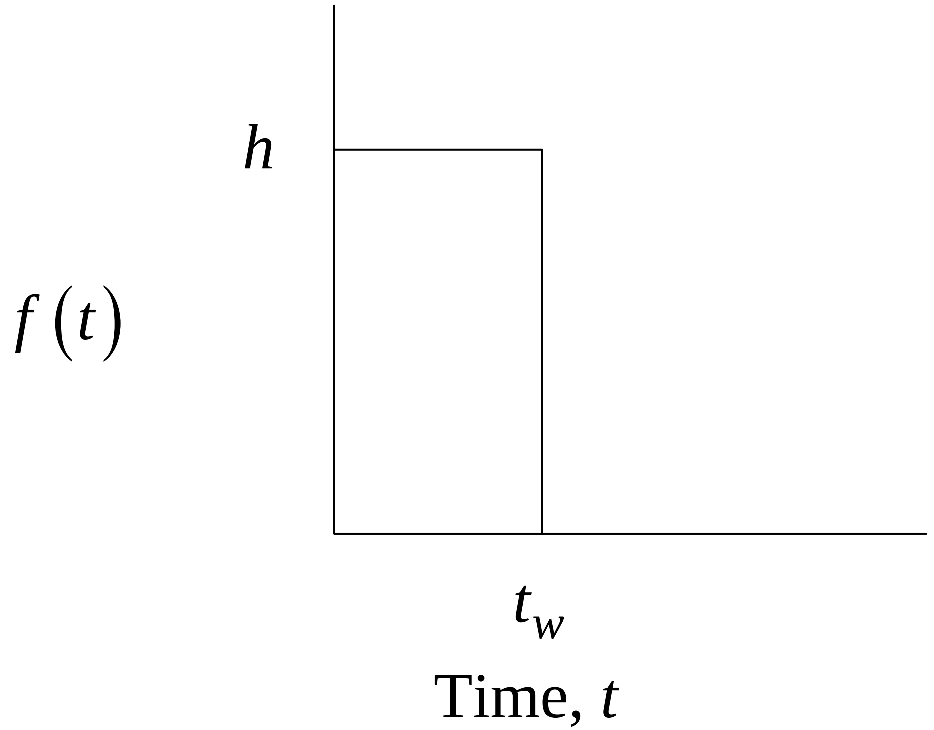
$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ h & \text{for } 0 \le t < t_w \\ 0 & \text{for } t \ge t_w \end{cases} \qquad (3\text{-}20)$$

6. Impulse Function (or Dirac Delta Function)

The impulse function is obtained by taking the limit of the rectangular pulse as its width, t_w, goes to zero but holding the area under the pulse constant at one. (i.e., let $h = \dfrac{1}{t_w}$)

Let, $\delta(t)$ □ impulse function

Then, $L\left[\delta(t)\right] = 1$



The Laplace transform of the rectangular pulse is given by

$$F(s) = \frac{h}{s}\left(1 - e^{-t_w s}\right) \qquad (3\text{-}22)$$

Other Transforms

Note:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-jwt} = \cos \omega t - j \sin \omega t$$

$$j = \sqrt{-1}$$

$$\mathbf{L}(\cos \omega t) = \mathbf{L}\left(\frac{e^{-j\omega t} + e^{+j\omega t}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s + j\omega} + \frac{1}{s - j\omega}\right)$$

$$= \frac{1}{2}\left(\frac{s - j\omega}{s^2 + \omega^2} + \frac{s + j\omega}{s^2 + \omega^2}\right)$$

$$= \frac{s}{s^2 + \omega^2}$$

$$\mathbf{L}(\sin \omega t) = \mathbf{L}\left(\frac{e^{+j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Difference of two step inputs S(t) – S(t-1)

(S(t-1) is step starting at t = h = 1)

By Laplace transform

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

Can be generalized to steps of different magnitudes (a_1, a_2).

Table 3.1. Laplace Transforms

See page 54 of the text.