Second-Order Systems

- Standard form:

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \tag{5-40}$$

which has three model parameters:

$K \triangleq$ steady-state gain

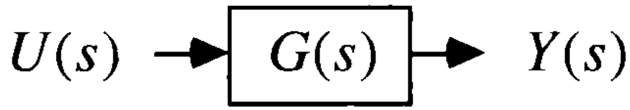
$\tau \triangleq$ "time constant" [=] time

$\zeta \triangleq$ damping coefficient (dimensionless)

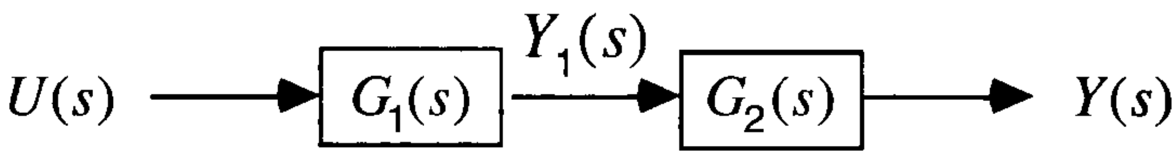
- Equivalent form: $\left(\omega_n \triangleq \text{natural frequency} = \dfrac{1}{\tau}\right)$

$$\frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Block Notation:



Composed of two first order subsystems (G_1 and G_2)



$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\tau = \frac{1}{\sqrt{\tau_1 \tau_2}}$$

$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}}$$

roots: $\dfrac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$

$\zeta > 1$ *overdamped*

$\zeta < 1$ *underdamped*

$\zeta = 1$ *critically damped*

2nd order ODE model (overdamped)

$$G(s) = \frac{K}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1}$$

- The type of behavior that occurs depends on the numerical value of damping coefficient, ζ :

It is convenient to consider three types of behavior:

Damping Coefficient	Type of Response	Roots of Charact. Polynomial
$\zeta > 1$	Overdamped	Real and $\neq$
$\zeta = 1$	Critically damped	Real and $=$
$0 \leq \zeta < 1$	Underdamped	Complex conjugates

- Note: The characteristic polynomial is the denominator of the transfer function:

$$\tau^2 s^2 + 2\zeta\tau s + 1$$

- What about $\zeta < 0$? It results in an unstable system

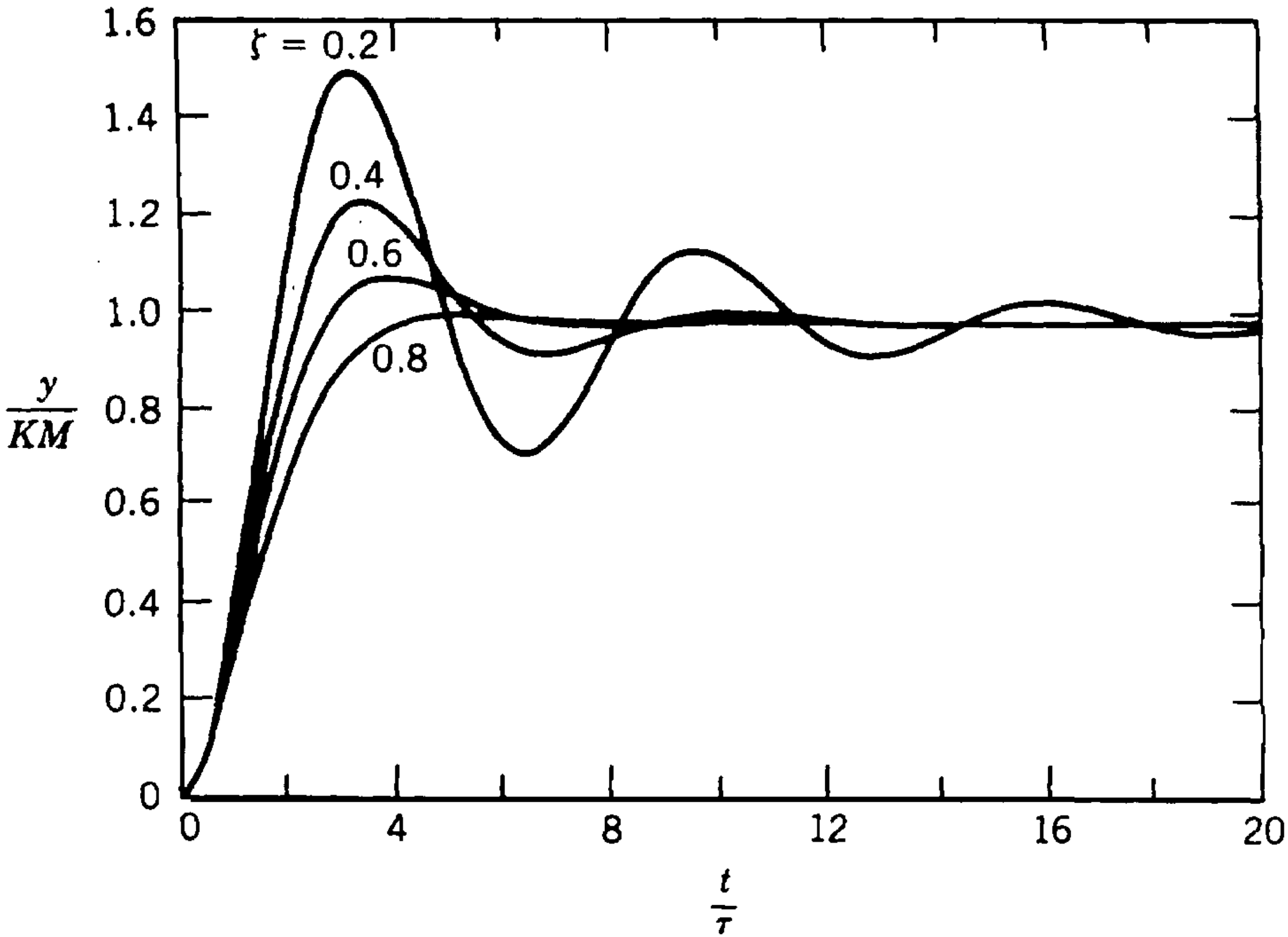


Figure 5.8. Step response of underdamped second-order processes.

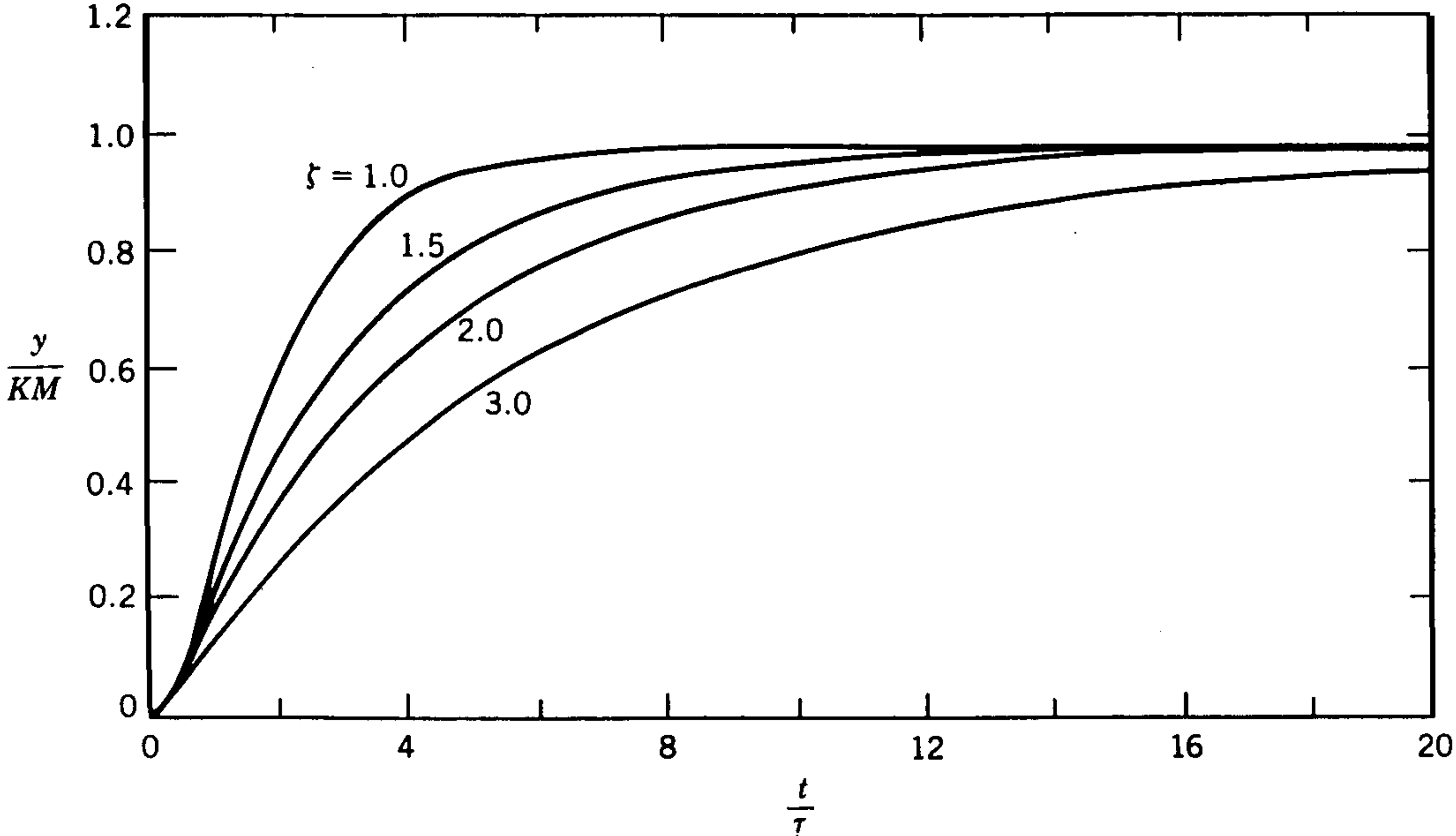


Figure 5.9. Step response of critically-damped and overdamped second-order processes.

Several general remarks can be made concerning the responses show in Figs. 5.8 and 5.9:

1. Responses exhibiting oscillation and overshoot ($y/KM > 1$) are obtained only for values of ζ less than one.
2. Large values of ζ yield a sluggish (slow) response.
3. The fastest response without overshoot is obtained for the critically damped case $(\zeta = 1)$.

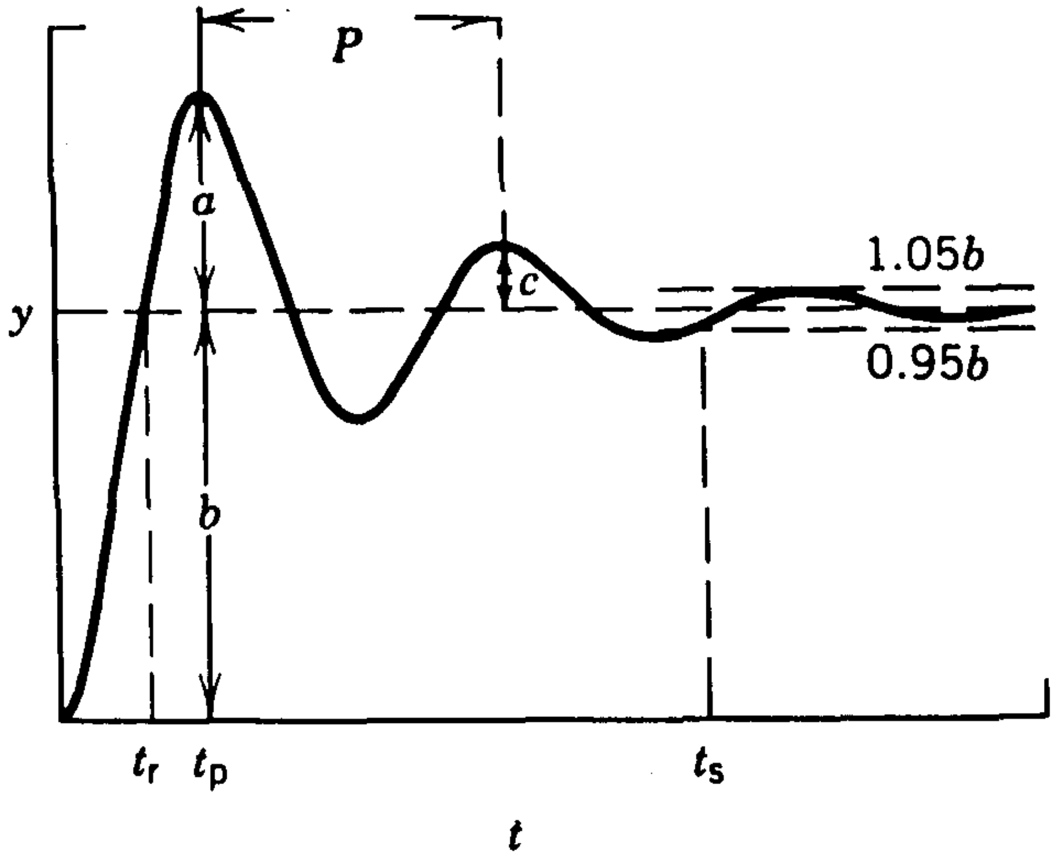


Figure 5.10. Performance characteristics for the step response of an underdamped process.

Chapter 5

1. **Rise Time:** t_r is the time the process output takes to first reach the new steady-state value.

2. **Time to First Peak:** t_p is the time required for the output to reach its first maximum value.

3. **Settling Time**: t_s is defined as the time required for the process output to reach and remain inside a band whose width is equal to ±5% of the total change in *y*. The term 95% response time sometimes is used to refer to this case. Also, values of ±1% sometimes are used.

4. **Overshoot:** OS = *a*/*b* (% overshoot is 100*a*/*b*).

5. **Decay Ratio**: DR = *c*/*a* (where *c* is the height of the second peak).

6. **Period of Oscillation**: *P* is the time between two successive peaks or two successive valleys of the response.

Chapter 5

Second Order Step Change

a. Overshoot

$$\frac{a}{b} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

b. time of first maximum

$$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

c. decay ratio (successive maxima – not min.)

$$\frac{c}{a} = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = \frac{a^2}{b^2}$$

d. period of oscillation

$$p = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

Sinusoidal

$$\rightarrow Y(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \cdot \frac{A\omega}{s^2+\omega^2}$$

$$\phi = -\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$$

$\mathcal{L}^{-1}$

$$y(t) = \frac{KA}{\{[1-(\omega\tau)^2]^2 + (2\zeta\omega\tau)^2\}^{0.5}} \sin(\omega t + \phi)$$

Input amplitude $= A$

Output " $= \hat{A} = \dfrac{KA}{\{[1-(\omega\tau)^2]^2 + (2\zeta\omega\tau)^2\}^{0.5}}$

amplitude ratio (AR) $= \dfrac{\hat{A}}{A}$

Normalized amplitude ratio (AR_N) $= \dfrac{\hat{A}}{KA}$

→ the effect of τ & ζ on the sinusoidal resp.

Max. AR_N : $\dfrac{\partial AR_N}{\partial \omega} = 0 \Rightarrow \omega_{max} = \dfrac{\sqrt{1-2\zeta^2}}{\tau}$ $\quad 0<\zeta<0.707$

Substitute $\omega_{max} \Rightarrow AR_N\Big|_{max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

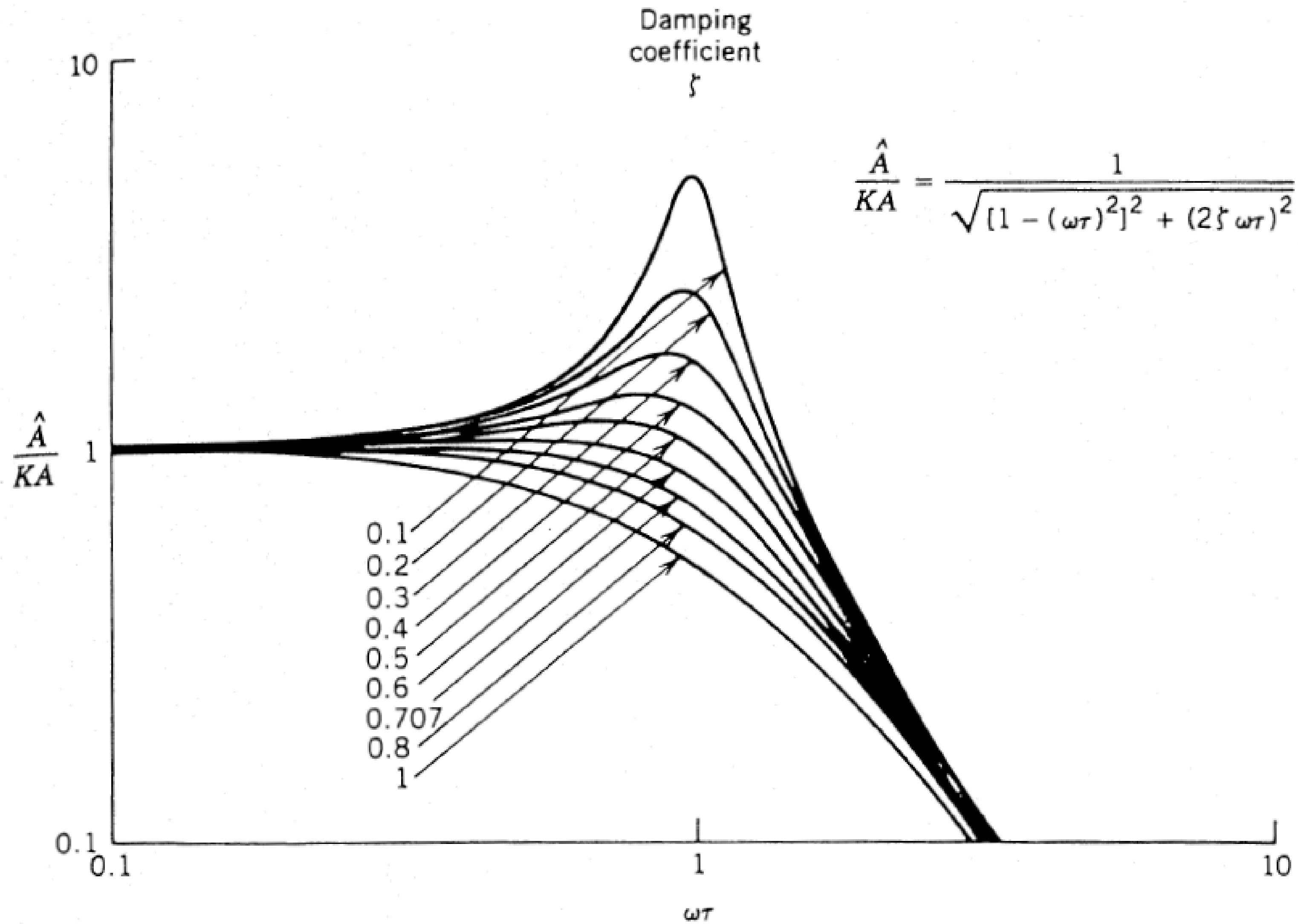


Figure 5.12. Sinusoidal response amplitude of a second-order system after exponential terms have become negligible.

Chapter 5

(a) $\frac{Y(s)}{X(s)} = \frac{18}{s^2+3s+9}$

After step change $3S(t)$
a) new s.s.?
b) for $y \leq 10$; what is max step change?

$x(t) = 3S(t)$, $X(s) = 3/s$

$$Y(s) = \frac{3 \times 18}{s(s^2+3s+9)}$$

$$y(t \to \infty) = \lim_{s \to 0} sY(s) = \frac{3 \times 18}{9} = 6$$

(a) $$\frac{Y(s)}{X(s)} = \frac{18}{s^2+3s+9}$$

After step change $3S(t)$
a) new s.s.?
b) for $y \leq 10$; what is max step change?

(b) $$\frac{Y(s)}{X(s)} = \frac{2}{\frac{1}{9}s^2+\frac{1}{3}s+1} = \frac{K}{\tau^2 s^2+2\zeta\tau s+1}$$

$$\tau^2 = \frac{1}{9}, \quad 2\zeta\tau = \frac{1}{3}, \quad K = 2$$

$$\tau = \frac{1}{3}, \quad \zeta = \frac{1}{2} < 1$$, system is underdamped

From equation (5-52),

$$\text{overshoot} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.163$$

For $x(t) = A\,S(t)$

$$y_{max} = KA + 0.163\,KA = 2.326\,A$$

For $y_{max} = 10$, $A = \frac{y_{max}}{2.326} = \frac{10}{2.326} = 4.3$

Largest step change in x is 4.3.