First-Order System

The standard form for a first-order TF is:

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \qquad (5\text{-}16)$$

where:

$K \triangleq$ steady-state gain

$\tau \triangleq$ time constant

Consider the response of this system to a step of magnitude, *M*:

$$U(t) = M \text{ for } t \geq 0 \qquad \Rightarrow U(s) = \frac{M}{s}$$

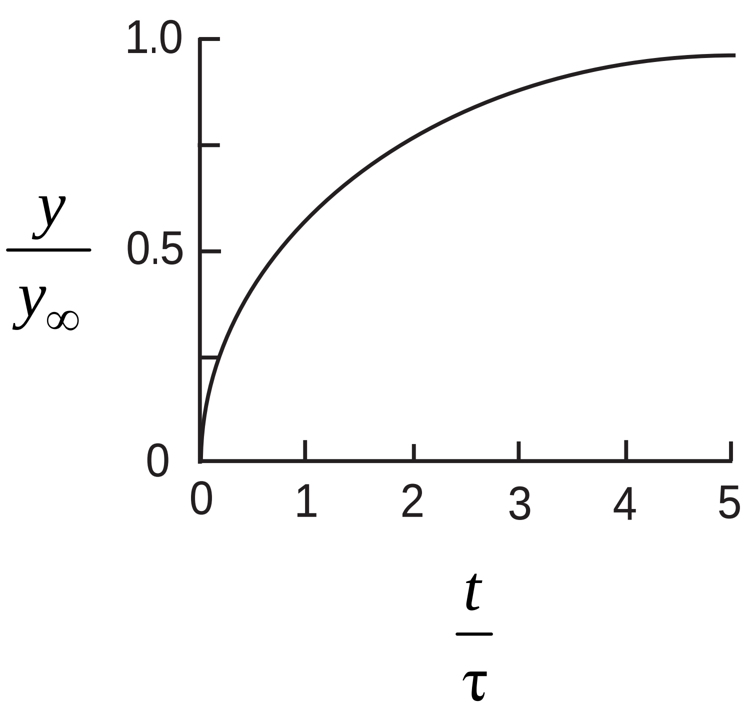
Substitute into (5-16) and rearrange,

$$Y(s) = \frac{KM}{s(\tau s + 1)} \qquad (5\text{-}17)$$

Take L^{-1} (cf. Table 3.1),

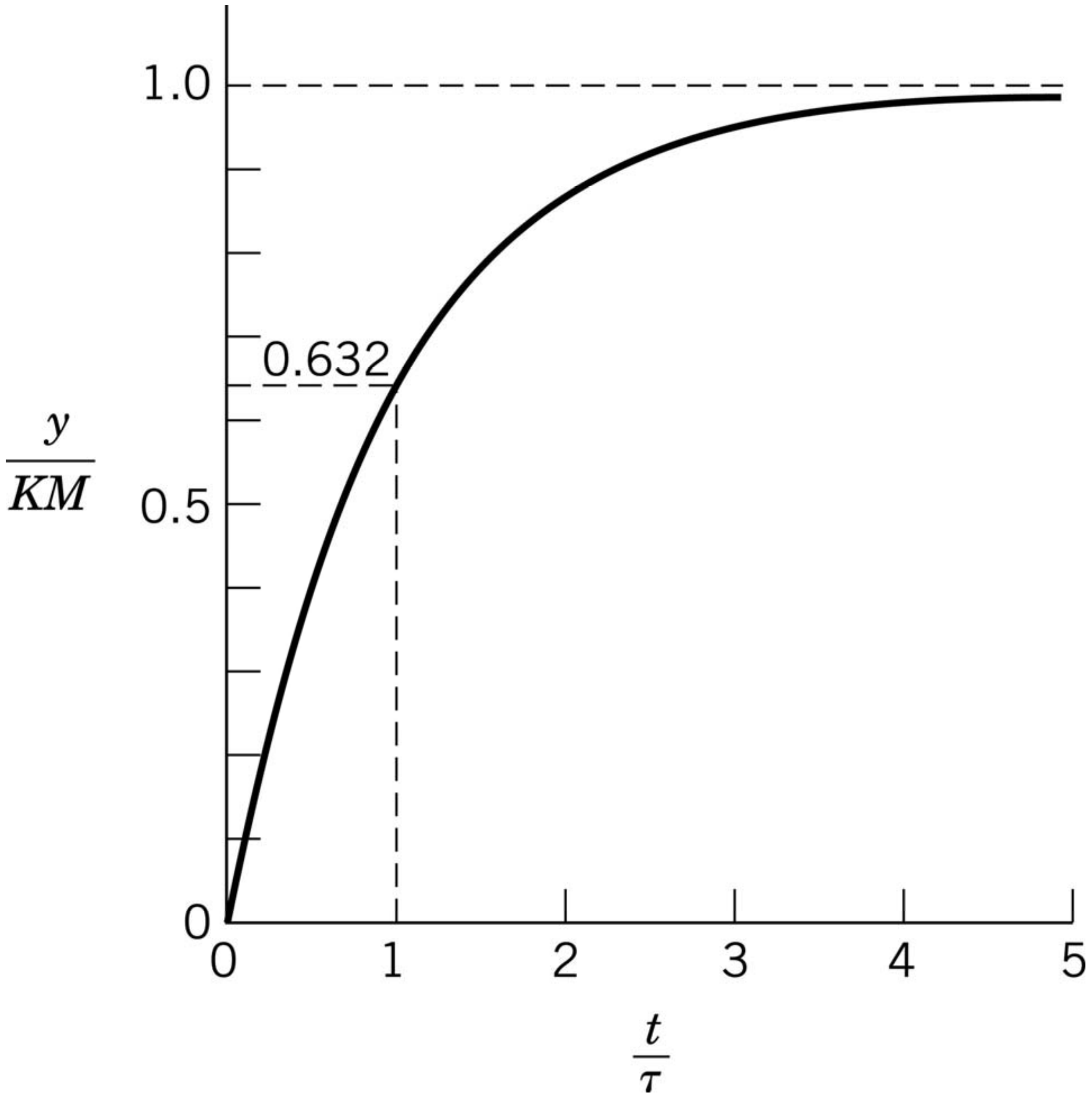
$$\boxed{y(t) = KM\left(1 - e^{-t/\tau}\right)} \qquad (5\text{-}18)$$

Let $y_\infty \equiv$ steady-state value of $y(t)$. From (5-18), $y_\infty = KM$.

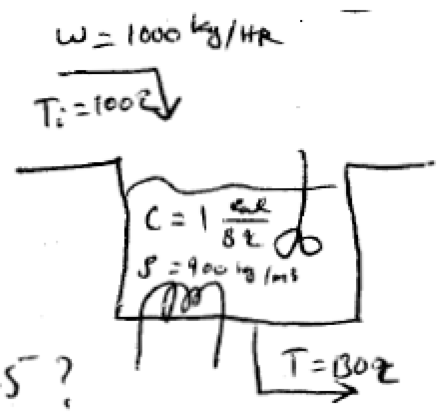


t	$\frac{y}{y_\infty}$
0	0
τ	0.632
2τ	0.865
3τ	0.950
4τ	0.982
5τ	0.993

Note: Large τ means a slow response.



1.0
0.632
$\frac{y}{KM}$
0.5
0
0
1
2
3
4
5
$\frac{t}{\tau}$



Ex

① Steady state Q ?

② +30% of Q, how long to get to SS ?

③ +20°C of T_i ; how long for T to reach 135 ?

① $\bar{Q} = \bar{w}\, c\, (\bar{T} - \bar{T}_i) = 3 \times 10^{7}\ \frac{cal}{HR}$

② $K = \frac{1}{wc} = 10^{-6}\ \frac{°C}{cal/HR}$ & $\tau = \frac{V\rho}{w} = 1.8$ hr

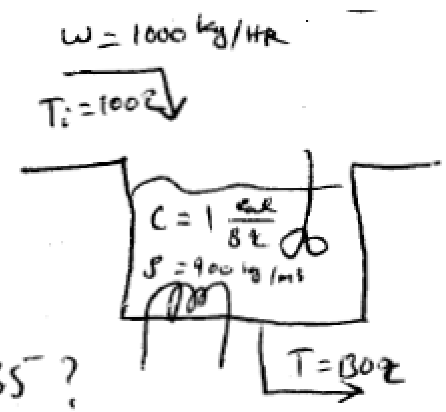
to reach SS it takes $5\tau = 9$ hr

also, F.V.T. $\lim_{s \to 0} s \frac{K}{\tau s + 1} \frac{0.3\bar{Q}}{s} = 9°C$

$\Rightarrow$ final $T = 130 + 9 = 139°C$ after +30% in Q

Or by looking into gain: +30% in Q $\rightarrow$ +0.3Q K in out

<u>Ex</u>



① Steady state Q ?

② +30% of Q, how long to get to SS ?

③ + 20°C of T_i ; how long for T to reach 135 ?

③ by looking to gain : final T is 150

$$\Rightarrow \frac{135-130}{150-130} = \frac{5}{20} = 25\%$$

from fig 5.4 $\rightarrow \frac{t}{\tau} = 0.3 \Rightarrow t = 0.54$ hr

<u>OR</u> $\frac{y(t)}{KM} = 1 - e^{-t/\tau} \rightarrow \frac{5}{(1)(20)} = 1 - e^{-t/1.8}$

$\Rightarrow t = 0.52$ hr

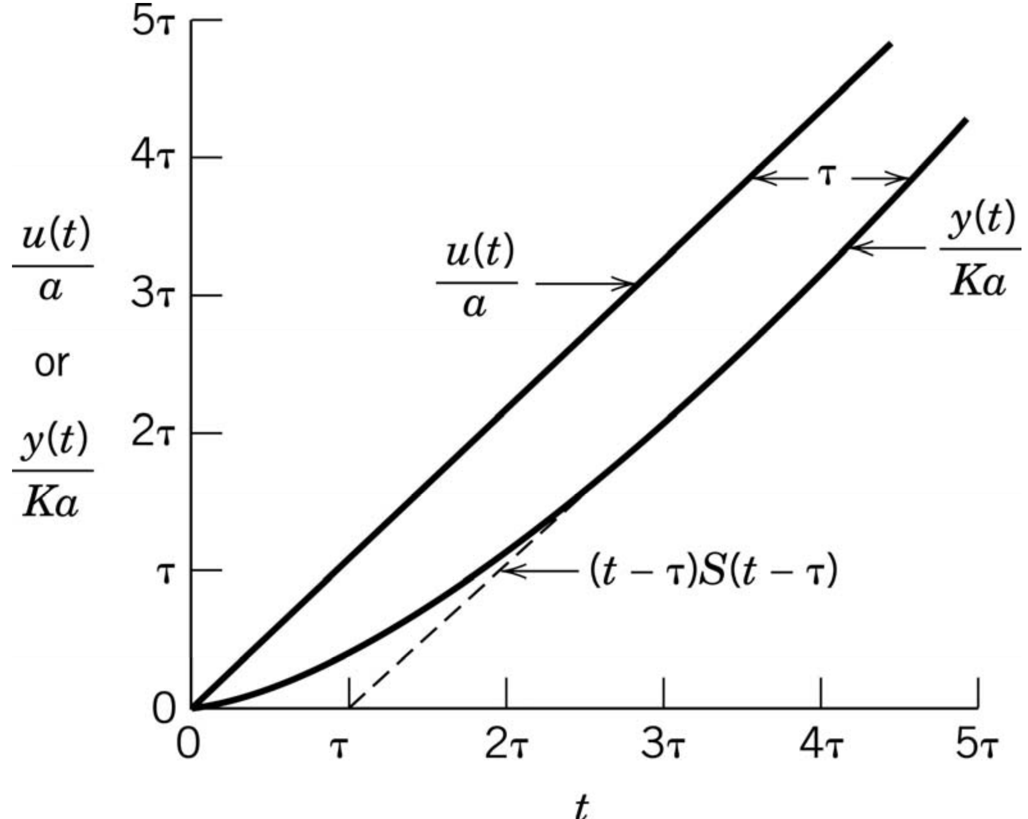
– Ramp

$$Y(s) = \frac{K}{\tau s+1}\,\frac{1}{s^2} = \frac{\alpha_1}{\tau s+1} + \frac{\alpha_2}{s} + \frac{\alpha_3}{s^2}$$

$$= \frac{Ka\tau^2}{\tau s+1} + \frac{Ka\tau}{s} + \frac{Ka}{s^2}$$

$$\mathcal{L}^{-1} \Rightarrow y(t) = Ka\tau(e^{-t/\tau} - 1) + Kat$$

@ $t \gg \tau$ $\quad y(t) = Ka(t - \tau)$



- Sinusoidal Response: $Y(s) = \frac{K}{\tau s+1}\frac{A\omega}{s^2+\omega^2}$

$$Y(s) = \frac{KA}{\omega^2\tau^2+1}\left(\frac{\omega\tau^2}{\tau s+1} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right)$$

$\mathcal{L}^{-1}$

$$\rightarrow y(t) = \frac{KA}{\omega^2\tau^2+1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos\omega t + \sin\omega t\right)$$

- fluctuation of flow around a value. 

For a sine input (1st order process)

$$U(s) = \frac{\omega}{s^2 + \omega^2}$$

output is...

$$Y(s) = \frac{K_p}{\tau s + 1} \cdot \frac{\omega}{s^2 + \omega^2} = \frac{\alpha_0}{\tau s + 1} + \frac{\alpha_1 s}{s^2 + \omega^2} + \frac{\alpha_2}{s^2 + \omega^2}$$

By partial fraction decomposition,

$$\alpha_0 = \frac{\omega K_p \tau^2}{\omega^2 \tau^2 + 1}$$

$$\alpha_1 = \frac{-\omega K_p \tau}{\omega^2 \tau^2 + 1}$$

$$\alpha_2 = \frac{\omega K_p}{\omega^2 \tau^2 + 1}$$

Inverting,

this term dies out for large t

$$y(t) = \frac{K_p \omega\tau}{\omega^2\tau^2 + 1} e^{-t/\tau} + \frac{K_p}{\sqrt{\omega^2\tau^2 + 1}} \sin(\omega t + \phi)$$

$$\phi = -\arctan(\omega\tau)$$

note: ϕ is not a function of t but of τ and ω.

For large t, y(t) is also sinusoidal,
output sine is attenuated by…

$$\frac{1}{\sqrt{\omega^2\tau^2 + 1}}$$ (fast vs. slow ω)

Integrating Process

Not all processes have a steady-state gain. For example, an "integrating process" or "integrator" has the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{K}{s} \qquad (K = \text{constant})$$

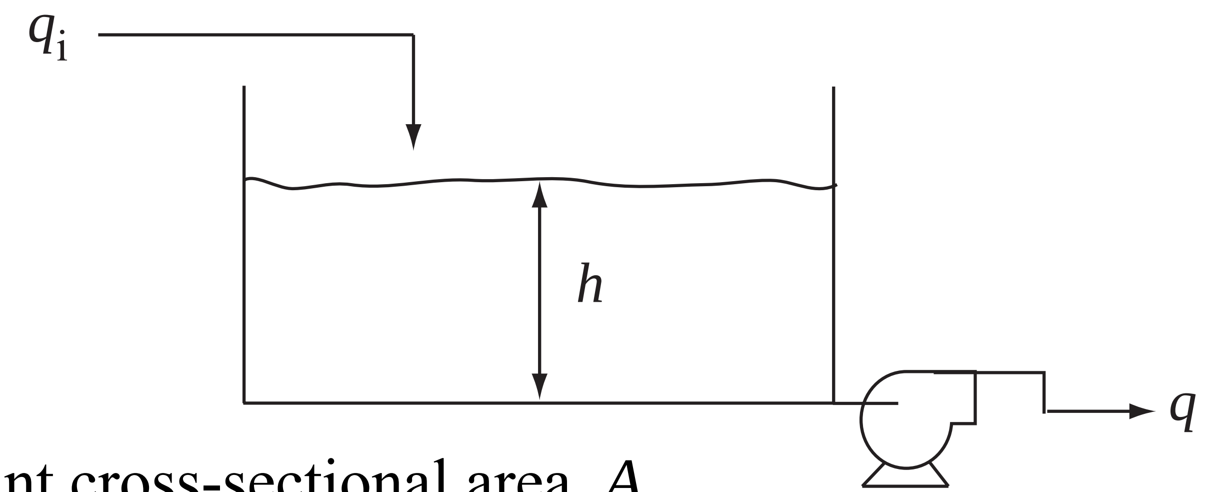
Consider a step change of magnitude *M*. Then $U(s) = M/s$ and,

$$Y(s) = \frac{KM}{s^2} \overset{\mathcal{L}^{-1}}{\Rightarrow} y(t) = KMt$$

Thus, $y(t)$ is unbounded and a new steady-state value does *not* exist.

Common Physical Example:

Consider a liquid storage tank with a pump on the exit line:



- Assume:
  1. Constant cross-sectional area, A.
  2. $q \neq f(h)$
- Mass balance: $A\dfrac{dh}{dt} = q_i - q$ (1) $\Rightarrow$ $0 = \bar{q}_i - \bar{q}$ (2)
- Eq. (1) – Eq. (2), take L, assume steady state initially,

$$H'(s) = \frac{1}{As}\left[Q_i'(s) - Q'(s)\right]$$

- For $Q'(s) = 0$ (constant q),

$$\boxed{\frac{H'(s)}{Q_i'(s)} = \frac{1}{As}}$$