Linearization of Nonlinear Models

Chapter 4

- So far, we have emphasized linear models which can be transformed into TF models.
- But most physical processes and physical models are nonlinear.
  - But over a small range of operating conditions, the behavior may be approximately linear.
  - *Conclude*: Linear approximations can be useful, especially for purpose of analysis.
- Approximate linear models can be obtained analytically by a method called "linearization". It is based on a Taylor Series Expansion of a nonlinear function about a specified operating point.

- Consider a nonlinear, dynamic model relating two process variables, *u* and *y*:

$$\frac{dy}{dt} = f(y,u) \qquad (4\text{-}60)$$

Perform a Taylor Series Expansion about $u = \bar{u}$ and $y = \bar{y}$ and truncate after the first order terms,

$$f(u,y) = f(\bar{u},\bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y' \qquad (4\text{-}61)$$

where $u' = u - \bar{u}$ and $y' = y - \bar{y}$. Note that the partial derivative terms are actually constants because they have been evaluated at the nominal operating point, $(\bar{u},\bar{y})$.

Substitute (4-61) into (4-60) gives:

$$\frac{dy}{dt} = f\left(\bar{u},\bar{y}\right) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y'$$

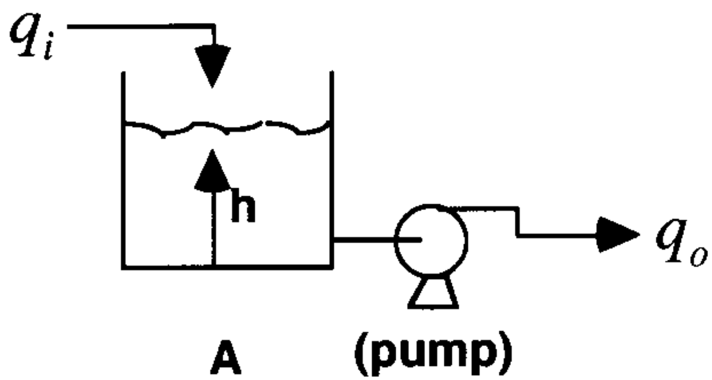
The steady-state version of (4-60) is:

$$0 = f\left(\bar{u},\bar{y}\right)$$

Substitute above and recall that $\frac{dy}{dt} = \frac{dy'}{dt}$,

$$\boxed{\frac{dy'}{dt} = \left.\frac{\partial f}{\partial u}\right|_{\bar{y}} u' + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y'} \qquad (4\text{-}62)$$

← Linearized model



q_0: control,
q_i: disturbance

$$A\frac{dh}{dt} = q_i - q_0$$

Use L.T. $AsH(s) = q_i(s) - q_0(s)$ (deviations)

$$q_o = \frac{1}{R_V} h$$

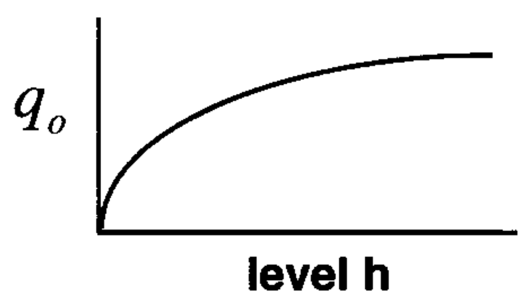
$$A\frac{dh}{dt} = q_i - \frac{1}{R_V} h$$

linear ODE : eq. (4-74)

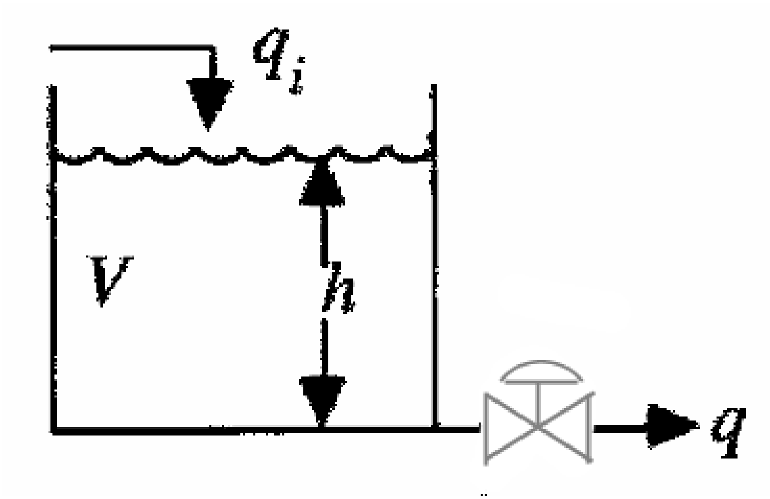
More realistically, if q_0 is manipulated by a flow control valve,

$$q_0 = C_v \sqrt{h}$$

nonlinear element



Example: Liquid Storage System



Mass balance: $$A\frac{dh}{dt} = q_i - q \quad (1)$$

Valve relation: $$q = C_v\sqrt{h} \quad (2)$$

A = area, C_v = constant

Combine (1) and (2),

$$A\frac{dh}{dt} = q_i - C_v\sqrt{h} \qquad (3)$$

Linearize $\sqrt{\ }$ term,

$$\sqrt{h} \approx \sqrt{\bar{h}} - \frac{1}{2\sqrt{\bar{h}}}\left(h - \bar{h}\right) \qquad (4)$$

Or

$$\sqrt{h} \approx \sqrt{\bar{h}} - \frac{1}{R}h' \qquad (5)$$

where:

$$R \triangleq 2\sqrt{\bar{h}}$$

$$h' \triangleq h - \bar{h}$$

Substitute linearized expression (5) into (3):

$$A\frac{dh}{dt} = q_i - C_v\left(\sqrt{\bar{h}} - \frac{1}{R}h'\right) \qquad (6)$$

The steady-state version of (3) is:

$$0 = \bar{q}_i - C_v\sqrt{\bar{h}} \qquad (7)$$

Subtract (7) from (6) and let $q_i' \triangleq q_i - \bar{q}_i$, noting that $\frac{dh}{dt} = \frac{dh'}{dt}$ gives the linearized model:

$$A\frac{dh'}{dt} = q_i' - \frac{1}{R}h' \qquad (8)$$

Summary:

In order to linearize a nonlinear, dynamic model:

1. Perform a Taylor Series Expansion of each nonlinear term and truncate after the first-order terms.
2. Subtract the steady-state version of the equation.
3. Introduce deviation variables.

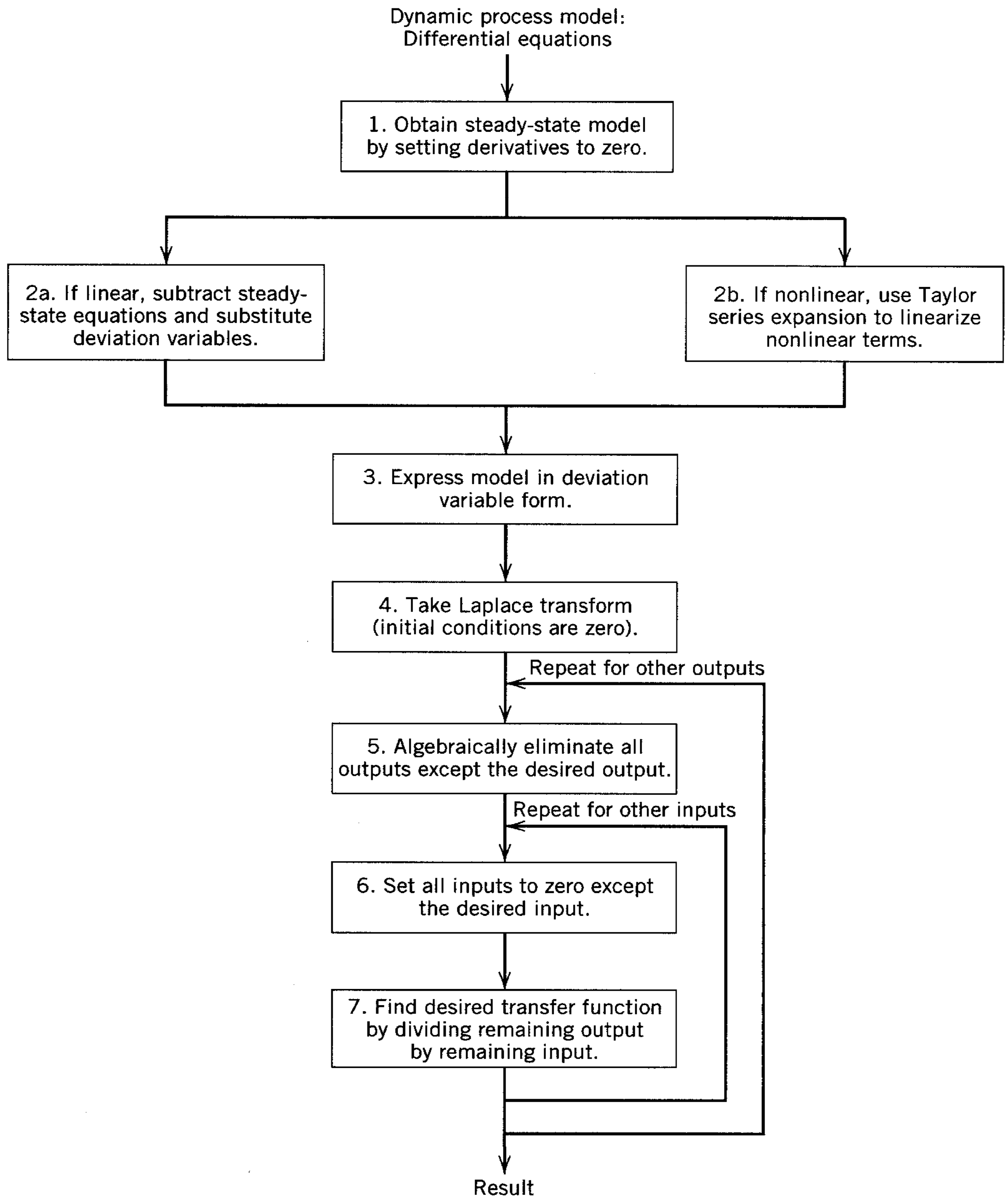


Figure 4.5 Procedure for developing transfer function models.

Solve Example 4.5, 4.6, 4.7

and

Solve Example 4.8

if you have any question ask me !

State-Space Models

- Dynamic models derived from physical principles typically consist of one or more ordinary differential equations (ODEs).
- In this section, we consider a general class of ODE models referred to as *state-space models*.

- Consider standard form for a *linear state-space model*,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}\boldsymbol{d} \qquad (4\text{-}90)$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \qquad (4\text{-}91)$$

where:

$\boldsymbol{x}$ = the *state vector*

$\boldsymbol{u}$ = the control *vector* of manipulated variables (also called *control variables*)

$\boldsymbol{d}$ = the disturbance vector

$\boldsymbol{y}$ = the *output vector* of measured variables. (We use boldface symbols to denote vector and matrices, and plain text to represent scalars.)

- The elements of $\boldsymbol{x}$ are referred to as *state variables*. WHY?
- The elements of $\boldsymbol{y}$ are typically a subset of $\boldsymbol{x}$, namely, the state variables that are measured. In general, $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{d}$, and $\boldsymbol{y}$ are functions of time.
- The time derivative of $\boldsymbol{x}$ is denoted by $\dot{\boldsymbol{x}}\left(= \mathrm{d}\boldsymbol{x}/\mathrm{d}t\right)$.
- Matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{E}$ are constant matrices.

Example 4.9

Show that the linearized CSTR model of Example 4.8 can be written in the state-space form of Eqs. 4-90 and 4-91. Derive state-space models for two cases:

(a) Both c_A and T are measured.

(b) Only T is measured.

Solution

The linearized CSTR model in Eqs. 4-84 and 4-85 can be written in vector-matrix form:

$$\begin{bmatrix} \dfrac{dc'_A}{dt} \\ \dfrac{dT'}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c'_A \\ T' \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} T'_s \qquad (4\text{-}92)$$

Let x_1 □ c'_A and x_2 □ T', and denote their time derivatives by $\dot{x}_1$ and $\dot{x}_2$. Suppose that the steam temperature T_s can be manipulated. For this situation, there is a scalar control variable, u □ T'_s , and no modeled disturbance. Substituting these definitions into (4-92) gives,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b_2 \end{bmatrix}}_{\boldsymbol{B}} u \qquad (4\text{-}93)$$

which is in the form of Eq. 4-90 with $\boldsymbol{x} = col\ [x_1, x_2]$. (The symbol "*col*" denotes a column vector.)

a) If both T and c_A are measured, then $\boldsymbol{y} = \boldsymbol{x}$, and $\boldsymbol{C} = \boldsymbol{I}$ in Eq. 4-91, where $\boldsymbol{I}$ denotes the 2x2 identity matrix. $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined in (4-93).

b) When only T is measured, output vector $\boldsymbol{y}$ is a scalar, $y = T'$ and $\boldsymbol{C}$ is a row vector, $\boldsymbol{C} = [0,1]$.

Note that the state-space model for Example 4.9 has $\boldsymbol{d} = \boldsymbol{0}$ because disturbance variables were not included in (4-92).

By contrast, suppose that the feed composition and feed temperature are considered to be disturbance variables in the original nonlinear CSTR model in Eqs. 2-60 and 2-64.

Then the linearized model would include two additional deviation variables, T_i' and c'_{Ai}

Chapter 4

Stability of State-Space Models

- The model will exhibit a bounded response ***x(t)*** for all bounded ***u(t)*** and ***d(t)*** if and only if the eigenvalues of A have negative real roots
- Solve example 4.10

Relationship between SS and TF

- $G_p(s) = C\,[sI-A]^{-1}\,B$
- $G_d(s) = C\,[sI-A]^{-1}\,E$
- Solve example 4.11