# Important Properties of Laplace Transforms

*1. Final Value Theorem*

It can be used to find the steady-state value of a closed loop system (providing that a steady-state value exists.

**Statement of FVT:**

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[ sY(s) \right]$$

providing that the limit exists (is finite) for all $\mathrm{Re}(s) \geq 0,$ where Re ($s$) denotes the real part of complex variable, $s$.

*2. Initial Value Theorem*

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*Example:*

Suppose,

$$Y(s) = \frac{5s+2}{s(5s+4)} \qquad (3\text{-}34)$$

Then,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\frac{5s+2}{5s+4}\right] = 0.5$$

3. *Time Delay*

Time delays occur due to fluid flow, time required to do an analysis (e.g., gas chromatograph). The delayed signal can be represented as

$$y(t-\theta) \qquad \theta = \text{time delay}$$

Also,

$$\mathcal{L}\left[y(t-\theta)\right] = e^{-\theta s}Y(s)$$

# Transfer Functions

- Convenient representation of a *linear*, dynamic model.
- A transfer function (TF) relates *one* input and *one* output:

$$\begin{matrix} x(t) \\ X(s) \end{matrix} \rightarrow \boxed{\text{system}} \rightarrow \begin{matrix} y(t) \\ Y(s) \end{matrix}$$

The following terminology is used:

| $\underline{x}$ | $\underline{y}$ |
|---|---|
| input | output |
| forcing function | response |
| “cause” | “effect” |

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## Definition of the transfer function:

Let $G(s)$ denote the transfer function between an input, $x$, and an output, $y$. Then, by definition

$$G(s) \triangleq \frac{Y(s)}{X(s)}$$

where:

$$Y(s) \triangleq \mathcal{L}\left[y(t)\right]$$

$$X(s) \triangleq \mathcal{L}\left[x(t)\right]$$

# Development of Transfer Functions

## Example: Stirred Tank Heating System

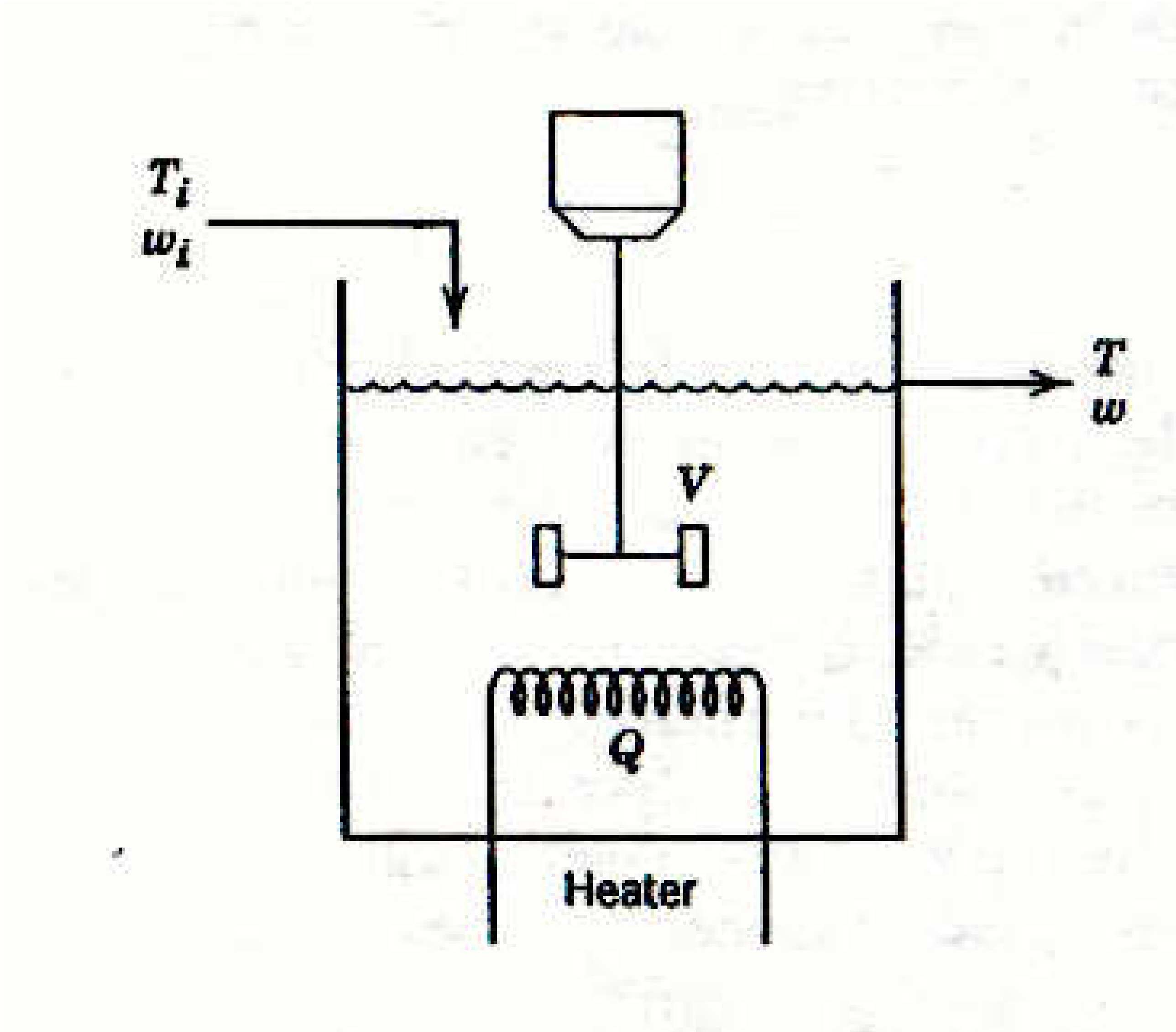


Figure 2.3 Stirred-tank heating process with constant holdup, *V*.

Recall the previous dynamic model, assuming constant liquid holdup and flow rates:

$$V\rho C\frac{dT}{dt} = wC\left(T_i - T\right) + Q \qquad (1)$$

Suppose the process is initially at steady state:

$$T\left(0\right) = \bar{T},\ \ T_i\left(0\right) = \bar{T}_i,\ \ Q\left(0\right) = \bar{Q} \qquad \left(2\right)$$

where $\bar{T}$ □ steady-state value of $T$, etc. For steady-state conditions:

$$0 = wC\left(\bar{T}_i - \bar{T}\right) + \bar{Q} \qquad (3)$$

Subtract (3) from (1):

$$V\rho C\frac{dT}{dt} = wC\left[\left(T_i - \bar{T}_i\right) - \left(T - \bar{T}\right)\right] + \left(Q - \bar{Q}\right) \qquad (4)$$

But,

$$\frac{dT}{dt} = \frac{d\left(T - \bar{T}\right)}{dt} \quad \text{because } \bar{T} \text{ is a constant} \tag{5}$$

Thus we can substitute into (4-2) to get,

$$V\rho C \frac{dT'}{dt} = wC\left(T_i' - T'\right) + Q' \tag{6}$$

where we have introduced the following "*deviation variables*", also called "perturbation variables":

$$T' \square\, T - \bar{T}, \quad T_i' \square\, T_i - \bar{T}_i, \quad Q' \square\, Q - \bar{Q} \tag{7}$$

Take L of (6):

$$V\rho C\left[sT'(s) - T'(t=0)\right] = wC\left[T_i'(s) - T'(s)\right] - Q'(s) \tag{8}$$

Evaluate $T'(t=0)$.

By definition, $T' \triangleq T - \bar{T}$. Thus at time, $t = 0$,

$$T'(0) = T(0) - \bar{T} \qquad (9)$$

But since our assumed initial condition was that the process was initially at steady state, i.e., $T(0) = \bar{T}$ it follows from (9) that $T'(0) = 0$.

<u>Note</u>: The advantage of using deviation variables is that the initial condition term becomes zero. This simplifies the later analysis.

Rearrange (8) to solve for $T'(s)$:

$$T'(s) = \left(\frac{K}{\tau s + 1}\right) Q'(s) + \left(\frac{1}{\tau s + 1}\right) T_i'(s) \qquad (10)$$

where two new symbols are defined:

$$K \triangleq \frac{1}{wC} \quad \text{and} \quad \tau \triangleq \frac{V\rho}{w} \qquad (11)$$

$$T'(s) = G_1(s)Q'(s) + G_2(s)T_i'(s)$$

$G_1$ and $G_2$ are transfer functions and independent of the inputs, Q′ and $T_i'$.
Note $G_1$ (process) has gain K and time constant $\tau$.

$G_2$ (disturbance) has gain=1 and time constant $\tau$.
gain = G(s=0). Both are first order processes.

If there is no change in inlet temperature ($T_i' = 0$), then $T_i'(s) = 0$.

System can be forced by a change in either $T_i$ or Q (see Example 4.1).

# Transfer Function Between $Q'$ and $T'$

Suppose $T_i$ is constant at the steady-state value. Then,

$T_i(t) = \bar{T}_i \Rightarrow T_i'(t) = 0 \Rightarrow T_i'(s) = 0.$ Then we can substitute into (10) and rearrange to get the desired TF:

$$\boxed{\frac{T'(s)}{Q'(s)} = \frac{K}{\tau s + 1}} \qquad (12)$$

# Transfer Function Between $T'$ and $T_i'$:

Suppose that $Q$ is constant at its steady-state value:

$$Q(t) = \bar{Q} \Rightarrow Q'(t) = 0 \Rightarrow Q'(s) = 0$$

Thus, rearranging

$$\boxed{\frac{T'(s)}{T_i'(s)} = \frac{1}{\tau s + 1}} \qquad (13)$$

Comments:

1. The TFs in (12) and (13) show the *individual* effects of $Q$ and $T_i$ on $T$. What about *simultaneous* changes in both $Q$ and $T_i$?

   - Answer: See (10). The same TFs are valid for simultaneous changes.

   - Note that (10) shows that the effects of changes in both $Q$ and $T_i$ are *additive*. This always occurs for linear, dynamic models (like TFs) because the Principle of Superposition is valid.

2. The TF model enables us to determine the output response to any change in an input.

3. Use deviation variables to eliminate initial conditions for TF models.

# Example: Stirred Tank Heater

$$K = 0.05 \qquad \tau = 2.0$$

$$T' = \frac{0.05}{2s+1} Q'$$

No change in $T_i'$

Step change in Q(t): 1500 cal/sec 2000 cal/sec

$$Q' = \frac{500}{s}$$

$$T' = \frac{0.05}{2s+1} \frac{500}{s} = \frac{25}{s(2s+1)}$$

What is T′(t)?

$$T'(t) = 25[1 - e^{-t/\tau}] \longleftarrow T(s) = \frac{25}{s(\tau s + 1)}$$

$$T'(t) = 25[1 - e^{-t/2}]$$