II. Solving First Order Linear ODE by Integrating Factor

A **first order linear ODE** is any differential equation of the form

$$a(x)\,y' + b(x)\,y = c(x)$$

where $a(x) \neq 0$, $b(x)$, and $c(x)$ are functions of x.

Solution: Method of Integrating Factor

1. Divide through by $a(x)$ to get the first order ODE in *standard form*

$$y' + p(x)\,y = q(x)$$

where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$.

2. Define the *integrating factor* via the coefficient of y in the standard form ODE

$$\mu(x) = e^{\int p(x)\,dx}$$

3. Multiply both sides of the standard form ODE by the integrating factor

$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,q(x) \qquad (*)$$

and recognize that this equation can then be written

$$[\mu(x)\,y]' = \mu(x)\,q(x)$$

because by the product rule for derivatives

$$[\mu(x)\,y]' = \mu(x)\,y' + \mu'(x)y$$

and by the chain rule for derivatives

$$\begin{aligned}\mu'(x) &= \frac{d}{dx}\left[e^{\int p(x)\,dx}\right]\\ &= e^{\int p(x)\,dx}\,\frac{d}{dx}\left[\int p(x)\,dx\right]\\ &= \mu(x)\,p(x)\end{aligned}$$

Summarizing: the left hand side of equation $(*)$

$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,y' + \mu'(x)y = [\mu(x)\,y]'$$

4. Integrate both sides of

$$[\mu(x)\,y]' = \mu(x)\,q(x)$$

to get

$$\mu(x)\,y = \int \mu(x)\,q(x)\,dx$$

and divide through by $\mu(x)$ to arrive at the solution to the ODE

$$y = \frac{1}{\mu(x)} \int \mu(x)\,\frac{c(x)}{a(x)}\,dx$$

where the integrating factor $\mu(x) = e^{\int \frac{b(x)}{a(x)}\,dx}$.