Mechanical Energy Balance

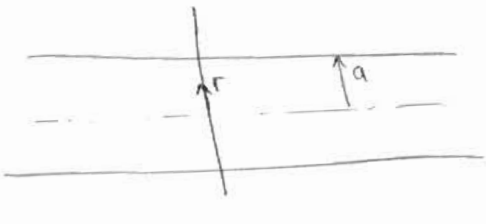
$$\Delta\left(\frac{u_m^2}{2}\right) + \Delta(gz) + \Delta\left(\frac{P}{\rho}\right) + F + W = 0$$

the term $\Delta\left(\frac{u_m^2}{2}\right) = \frac{u_2^2}{2} - \frac{u_1^2}{2}$ (u_1 and u_2 are mean or average velocities)

mean or average kinetic energy

for laminar flow

$$u = \frac{-\Delta P}{4\mu L}(a^2 - r^2) \qquad (1)$$



For any quantity ψ per unit mass

$$\text{average value} \quad \overline{\psi} = \frac{\int_0^a 2\pi r\,dr\,(\rho u \psi)}{\int_0^a 2\pi r\,dr\,(\rho u)} \qquad (2)$$

numerator → Total amount of ψ flowing

denominator → total mass flow rate

$$\overline{\psi} = \frac{u^2}{2} \quad \text{: kinetic energy/mass}$$

$$\overline{\frac{u^2}{2}} = \overline{KE} = u_m^2 \quad \text{not } \left(\frac{u_m^2}{2}\right)$$

if we substitute the velocity profile given by (1) in (2) and integrating

For turbulant flow u has different relation with r

$$\overline{K.E}\Big|_{turbulent} = \frac{u_m^2}{2} * (1 \text{ to } 1.07) \approx \frac{u_m^2}{2}$$

∴ The accurate mechanical energy balance equation should be given by

$$\alpha \Delta\left(\frac{u^2}{2}\right) + \Delta(gz) + \Delta\left(\frac{P}{\rho}\right) + F + W = 0$$

$\alpha = 2$ for laminar flow.

$\alpha = 1$ for turbulent flow.

Friction Factor (f)

Dimenssionless quantity given by:

$$f = \frac{\tau_w}{\rho u_m^2}$$

τ_w → wall shear stress

ρu_m^2 → inertial force per unit area that would result from the Impigement of a stream of density ρ and Velocity u_m normally against a wall

Other Common forms of friction factor

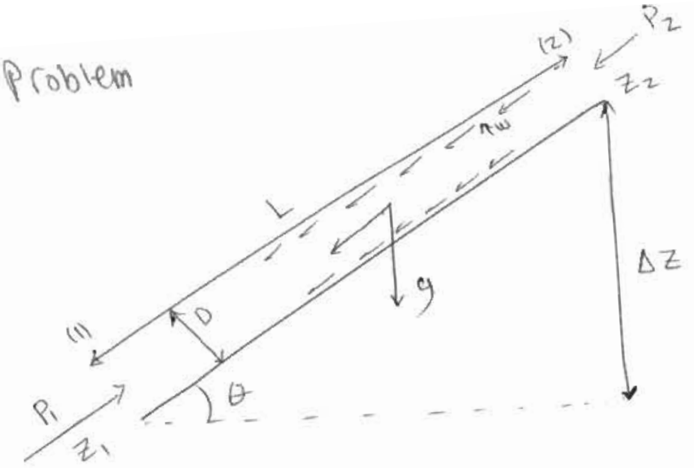
$$f_f = \frac{\tau_w}{\frac{1}{2}\rho u_m^2}$$ Fanning friction factor

$$f_m = \frac{\tau_w}{\frac{1}{8}\rho u_m^2}$$ mody friction factor

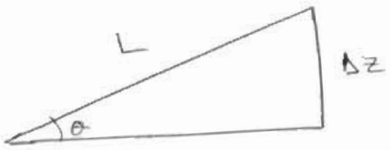
We will use f_F and ommit the subscript F

$$\therefore f = \frac{\tau_w}{\frac{1}{2}\rho u_m^2}$$

Piping and pumping Problem



Momentum Balance (Direction of flow ↗)



$\sin\theta = \frac{\Delta Z}{L}$ from geometry

$$\dot{\mathcal{M}}_{in} - \dot{\mathcal{M}}_{out} + \sum F = \frac{d\mathcal{M}}{dt}$$

$$\dot{\mathcal{M}}_{in} = m_1 u_1 \quad , \quad \dot{\mathcal{M}}_{out} = m_2 u_2$$

$m_1 = m_2$ (mass balance)

also, $u_1 = u_2$ (ρ and A are constants)

$$\dot{\mathcal{M}}_{in} = \dot{\mathcal{M}}_{out} = m u$$

$$\frac{d\mathcal{M}}{dt} = 0 \quad (S.S)$$

$$\therefore \sum F = 0$$

$$\therefore P_1 A - P_2 A - \tau_w (\pi D L) - M g \sin\theta = 0$$

$$(P_1 - P_2)\frac{\pi}{4} D^2 - \tau_w (\pi D L) - \frac{\pi}{4} D^2 L \rho g \sin\theta = 0$$

$A = \pi \frac{D^2}{4}$

$M = V\rho$
$= \frac{\pi}{4} D^2 L \rho$

$$\sin\theta = \frac{\Delta Z}{L} \Longrightarrow \Delta Z = L\sin\theta$$

$$P_1 - P_2 = -\Delta P$$

$$-\Delta P\left(\frac{\pi}{4}D^2\right) - \tau_w(\pi D L) - \frac{\pi}{4}D^2 g \rho \Delta Z = 0$$

$$\therefore \boxed{-\Delta P = P_1 - P_2 = \rho g \Delta Z + 4\tau_w\left(\frac{L}{D}\right)}$$

the same equation is obtained for downward flow

ΔZ is negative

$$f = \frac{\tau_w}{\frac{1}{2}\rho u_m^2}$$

$$\tau_w = \tfrac{1}{2}\rho u_m^2 f$$

$$\therefore -\Delta P = \rho g \Delta Z + 2\rho u_m^2 f\left(\frac{L}{D}\right) \qquad (*)$$

The pressure drop depend on friction and gravity

$$\text{M.E.B}: \quad \underbrace{\alpha\rho\left(\Delta\left[\frac{u^2}{2}\right]\right)}_{0} + \rho g\Delta Z + \Delta P + \rho F + \underbrace{\rho w}_{0} = 0$$

$$\therefore -\Delta P = \rho g \Delta Z + \rho F \qquad (**)$$

compare $(*)$ & $(**)$

$$\Longrightarrow F = 2 * u_m^2 * f * \left(\frac{L}{D}\right)$$

$$Q = u_m\left(\frac{\pi}{4}\right)D^2 \Longrightarrow Q^2 = u_m^2\left(\frac{\pi}{4}\right)^2 D^4$$

$$\text{or} \quad u_m^2 = \frac{Q^2}{\frac{\pi^2}{16}D^4}$$

$$F = 2 * \frac{Q^2}{\pi^2 D^4} * 16 * f * \left(\frac{L}{D}\right)$$

$$\therefore \boxed{F = \frac{32 f Q^2 L}{\pi^2 D^5}}$$

For laminar flow:

it was shown that $F = \frac{8 \mu U_m L}{\rho r^2}$

$$\therefore \frac{8 \mu U_m L}{\rho D^2/4} = 2 U_m^2 * f * \frac{L}{D}$$

$$\therefore f = \frac{32 \mu U_m L * D}{2 * \rho * D^2 * U_m^2 * L} = 16 * \frac{\mu}{\rho D U_m} \quad 1/Re$$

$$\therefore \boxed{f = \frac{16}{Re}} \quad \text{Laminar flow}$$

For turbulant flow:

f depends on Re and pipe roughness ε

ε: length scale assigned to a particular wall material

Values of ε for various surfaces is given in table 3.2

Cast Iron, $\varepsilon = 0.25$ mm

Commercial Steel $\varepsilon = 0.046$ mm

Figure 3.10 Fanning friction factor for flow in Pipes

* For laminar flow $f = \frac{16}{Re}$

* for turbulant flow

$$\frac{1}{\sqrt{f}} = -1.737 \ln\left(0.269\frac{\varepsilon}{D} + \frac{1.257}{Re\sqrt{f}}\right)$$

Colebrook-White equation

For smoth pipe, $Re < 100{,}000$

$$f = 0.0790\, Re^{-1/4}$$

for high Re f is independent of Re

Shacham equation:

$$f = \left\{-1.737 \ln\left[0.269\frac{\varepsilon}{D} - \frac{2.185}{Re}\ln\left(0.269\frac{\varepsilon}{D} + \frac{14.5}{Re}\right)\right]\right\}^{-2}$$

Pipe Commercial sizes

$$\text{schedule \# } n = 1000\,\frac{P_{max}}{S_a}$$

maximum alloable Pressure in the Pipe

allowable tensile Stress in the Pipe wall

see Table 3.3

Pressure drop across pipe fittings:

* Fittings such as valves and elbows are associated with most piping installations

* Fittings induce additional turbulence and frictional dissipation

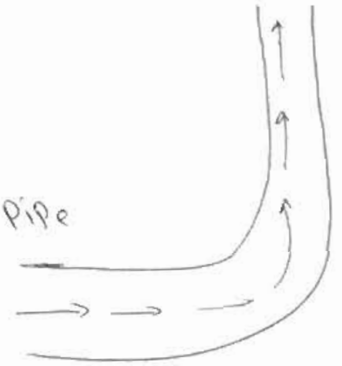
$\Downarrow$ Additional pressure drop

* Fitting causes an additional ΔP that would be produced by a certain length of pipe

* Effect of fitting on ΔP is correlated in terms of equivalent length.

Example:

90° elbow in a 6-in diameter line causes a pressure drop = 15 ft of pipe

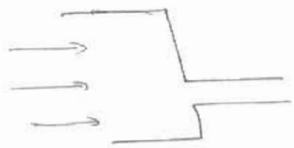


$$\left(\frac{L}{D}\right)_e = 30$$

$$\Rightarrow L = 30 * \frac{6}{12} = 15 \text{ ft}$$

Table 3.4: Type of fitting vs $\left(\frac{L}{D}\right)_e$

Fitting: Example



Sudden Contraction

Examples 3.2, 3.3 and 3.4

Unloading Oil from a

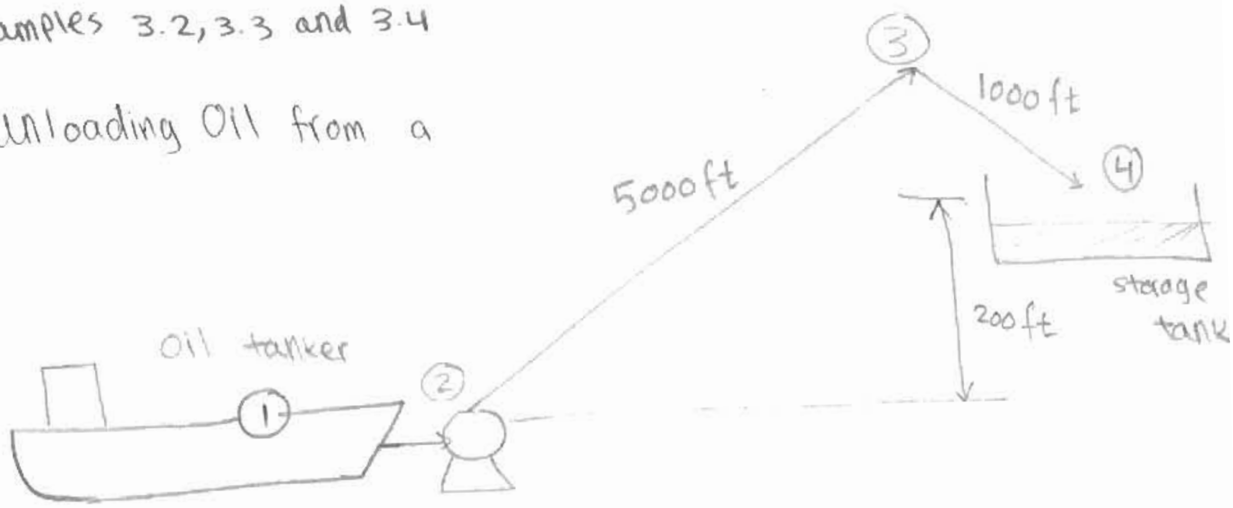


Fig E3.2

* fluid: Crude oil of 35°API / $\rho = 53 \frac{lbm}{ft^3}$, $\mu = 13.2\ cp$

$p^* = 4.0\ psi$

* Effective length of piping = 6000 ft including fittings
* Pipe type: Commercial Steel
* losses between point (1) and (2) may be ignored
* Free surfaces at (1) & (4) are open to the atmosphere.

Case (1) Flow rate and pipe diameter are given

D: nominal diameter of 6 in / Schedule 40.

Q: 506 gpm.

Find P_2: pressure at the pump exit.

Solution

$$Q = 506 \frac{gal}{min} * \frac{1 ft^3}{7.48 gal} * \frac{1 min}{60 s} = 1.127\ ft^3/s$$

$$D = 6.065\ in = 0.5054\ ft \quad (\text{Table 3.3})$$

$$Re = \frac{\rho U_m D}{\mu} = \frac{53 \frac{lbm}{ft^3} * 5.62\, ft/s * 0.5054\, ft}{13.2\, cP * 0.000672 \frac{lbm}{ft \cdot s}} = \boxed{16,790}$$

Turbulent

$$\frac{\varepsilon}{D} = \frac{0.00015}{0.5054}$$ → Table 3.2 Commercial steel

From shacham equation or Figure 3.10

$$f_F = \left\{ -1.737 \ln\left[0.269 \left(\frac{\varepsilon}{D}\right) - \frac{2.185}{Re} \ln\left(0.269 \left\{\frac{\varepsilon}{D}\right\} + \frac{14.5}{Re}\right)\right]\right\}^{-2}$$

$$f_F = 0.00692$$

$$F = 2 f_f * U_m^2 * \left(\frac{L}{D}\right)$$

M.E.B (2 & 4)

$$\frac{P_4 - P_2}{\rho} + \underset{\alpha}{1} * \frac{U_4^2 - U_2^2}{2} + g(Z_4 - Z_2) + F + \cancel{W}^{\,0} = 0$$

Material balance

$$U_4 = U_2$$

$$P_4 - P_2 = \rho g (Z_2 - Z_4) - \rho F$$

or $$P_2 = \cancel{P_4}^{\,0\ \text{gauge}} + \rho g (Z_4 - Z_2) + \rho F$$

$$P_2 = 53 \frac{lbm}{ft^3} * 32.2 \frac{ft}{s^2} * 200\, ft * \frac{1\, lbf}{32.2 \frac{lbm \cdot ft}{s^2}}$$

$$+ 53 \frac{lbm}{ft^3} * 2 * 0.00692 * (5.62)^2 \frac{ft^2}{s^2} * \left(\frac{6000}{0.5054}\right) * \frac{1\, lbf}{32.2 \frac{lbm \cdot ft}{s^2}}$$

$$P_2 = (10600 + 8542) \frac{lbf}{ft^2} * \left(\frac{1\, ft}{12\, in}\right)^2 = 132.9\, Psi$$

Case 2: Specified diameter and pressure drop

$P_2 = 132.7$ psig

$D = 0.5054$ ft (schedule 40 with a nominal diameter of 6in)

Find Q

Solution

$\frac{\varepsilon}{D} = 0.000297$ (same as case 1)

U_m can not be determined (Q is unknown)

$\Longrightarrow$ Re is unknown

Trial and error procedure:

1 * Assume Re^{old}, say $Re = 100,000$

2 * Evaluate f_F from Shacham equation

$$f_F = 0.00488 \quad \text{at} \quad Re = 100,000$$

3 * M.E.B (2 and 4) $\quad P_2 = \rho g (Z_4 - Z_2) + \rho \hat{F}$

or $P_2 = \rho g (Z_4 - Z_2) + 2\rho f_F U_m^2 * \left(\frac{L}{D}\right)$

$$\therefore U_m = \sqrt{\frac{D}{2\rho L f_F}\left[P_2 - \rho g (Z_4 - Z_2)\right]}$$

Convert this term to $\left(\frac{lbm \cdot ft}{s^2}\right)/ft^2$

$1\ lbf = 32.2\ \frac{lbm \cdot ft}{s^2}$

$\therefore U_m = 6.681$ ft/s

4 * Re evaluate Re $\Longrightarrow Re^{New} = 20,175$

5 * $\text{Diff} = \left| \dfrac{Re^{N} - Re^{old}}{Re^{N}} \right| * 100$

6 * If Diff < 1 % Stop and go to 7, else go to 2

7 * Compute $U_m = \dfrac{\mu Re^{N}}{\rho D}$, $Q = \frac{\pi}{4} D^2 U_m$

Since Diff $\not<$ 1%

repeat the calculations

Results

initial guess	Iteration	Re	U_m (ft/s)
→	0	100,000	6.681
	1	20,175	5.729
	2	17,301	5.630
	3	17,002	5.619
	4	$\boxed{16,968}$ good enough	5.617

$\therefore Q = \frac{\pi}{4} D^2 \left(\dfrac{\mu Re^{N}}{\rho D} \right) = 506 \text{ gal/min (gpm)}.$

Case 3: Specified Q and P_2 / unKown diameter

$Q = 506$ gal/min

$P_2 = 132.7$ Psig

$D = ?$

The Procedure is again trial and error

1* Guess D Say $D = 3$ in

2* Evaluate $U_m = \dfrac{Q}{\frac{\pi}{4} D^2}$

3* Evaluate Re

4* Evaluate $\dfrac{\varepsilon}{D}$

5* Find f_F from chart or shacham equation

6* Find D from $U_m = \sqrt{\dfrac{D}{2 f_F \rho L}\left[P_2 - g\Delta Z\right]}$

$$\therefore D^{New} = \frac{2 f_F \rho U_m^2 L}{P_2 - g\Delta Z}$$

7* Evaluate $Diff = \left|\dfrac{D_{New} - D_{old}}{D_{New}}\right| * 100$

8* If $Diff < 1$ stop, go to 9, else go to 2

9* $D = D^{New}$

Example 3.4—Unloading Oil from a Tanker Specified Flow Rate and Pressure Drop

Consider again the situation described at the beginning of Example 3.2, but now implement the algorithm for problems of type Case 3. If the flow rate is specified as $Q = 506$ gpm, and the available pressure at the pump exit is $p_2 = 132.7$ psig, what pipe diameter D (in.) is needed?

A hand calculation will not be presented here, for two reasons:

1. The reader should have understood the general idea from Cases 1 and 2, as exemplified by the detailed hand calculations in Examples 3.2 and 3.3.
2. More quantities are involved in the iterative calculations for Case 3, and there is much to be said for using spreadsheet calculations exclusively.

Therefore, the final spreadsheet solution is given in Table E3.4. Observe that there are now *five* mutually dependent quantities: D, u_m, Re, ε/D, and f_F. The Excel "iterate" feature is again used in order to converge rapidly on the final indicated values. □

Example 3.5—Unloading Oil from a Tanker Miscellaneous Additional Calculations

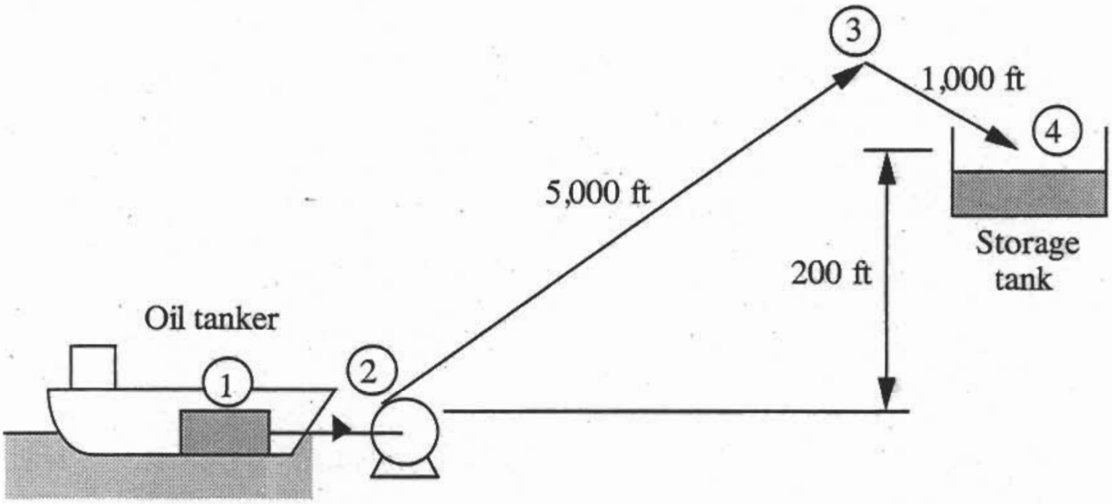


Fig. E3.5 (E3.2) Unloading tanker with intervening hill.

Still consider the situation studied in Examples 3.2, 3.3, and 3.4, for which $D = 0.5054$ ft, $p_2 = 132.7$ psig, and $Q = 506$ gpm. Answer the following additional questions:

(a) If the combination of pump and motor is 80% efficient, how much electrical power (kW) is needed to drive the pump?

(b) If, in order to avoid vapor-lock, the pressure in the pipeline must always be above the vapor pressure of the crude oil, what is the maximum permissible

Solution

(a) To obtain the necessary pumping power, first apply the energy balance across the pump, between points 1 and 2 (with no change in kinetic energy or elevation, and no explicit representation of friction):

$$\frac{p_2 - p_1}{\rho} + w = 0. \tag{E3.5.1}$$

The work performed *on* the crude oil, per unit mass, is:

$$-w = 132.7 \, \frac{\text{lb}_\text{f}}{\text{in}^2} \times 144 \, \frac{\text{in}^2}{\text{ft}^2} \times \frac{1}{53} \, \frac{\text{ft}^3}{\text{lb}_\text{m}} = 360 \, \frac{\text{ft lb}_\text{f}}{\text{lb}_\text{m}}. \tag{E3.5.2}$$

The mass flow rate of the oil is:

$$m = \frac{506 \times 53}{7.48 \times 60} = 59.75 \, \frac{\text{lb}_\text{m}}{\text{s}}. \tag{E3.5.3}$$

Bearing in mind the efficiency of 80%, the required electrical power to be delivered to the pump is:

$$P = \frac{360 \times 59.75}{737 \times 0.80} = 36.48 \text{ kW}. \tag{E3.5.4}$$

(b) Note that vapor-lock is most likely to occur at the highest elevation—namely, at point 3. Therefore, to find the maximum elevation of point 3 *without* causing vapor-lock, apply the energy equation between points 3 and 4, again with zero kinetic-energy change and no work term:

$$g(z_4 - z_3) + \frac{p_4 - p_3}{\rho} + 2 f_F u_m^2 \frac{L}{D} = 0, \tag{E3.5.5}$$

$$32.2(z_4 - z_3) + \underbrace{\frac{(14.7 - 4) \times 32.2 \times 144}{53}}_{936.1} + \underbrace{2 \times 0.00692 \times 5.617^2 \times \frac{1{,}000}{0.5054}}_{864.0} = 0. \tag{E3.5.6}$$

Solving for the elevation difference:

$$z_3 - z_4 = 55.9 \text{ ft}. \tag{E3.5.7}$$

That is, the highest point in the pipeline is limited to 55.9 ft elevation above the final discharge at point 4. If it were any higher, the pressure would fall to the vapor pressure of the oil (4.0 psia) and the oil would *start* to vaporize; the extent of vaporization would be limited by the amount of heat available to supply the necessary latent heat of vaporization.

(c) If the flow were at the upper limit of the laminar range, the Reynolds number would be $\text{Re} = \rho u_m D/\mu = 2{,}000$, corresponding to a mean velocity of:

$$u_m = \frac{2{,}000 \times 13.2 \times 0.000672}{53 \times 0.5054} = 0.662 \ \frac{\text{ft}}{\text{s}}. \tag{E3.5.8}$$

The corresponding frictional dissipation per unit mass is:

$$\mathcal{F} = \frac{8\mu LQ}{\pi a^4 \rho} = \frac{8\mu L u_m}{a^2 \rho} = \frac{8 \times 13.2 \times 0.000672 \times 6{,}000 \times 0.662}{\left(\frac{0.5054}{2}\right)^2 \times 53} = 83.32 \ \frac{\text{ft}^2}{\text{s}^2}. \tag{E3.5.9}$$

Application of the energy balance between points 2 and 4, with $p_4 = 0$ and $\Delta z =$ 200 ft, gives:

$$-\frac{p_2 \times 144 \times 32.2}{53} + 200 \times 32.2 + 83.32 = 0, \tag{E3.5.10}$$

so that the required pump exit pressure is:

$$p_2 = 74.6 \text{ psig}. \tag{E3.5.11}$$ □

Alternative treatment as simultaneous nonlinear equations. A different but equivalent approach to simple piping problems of the nature just discussed in Examples 3.2, 3.3, and 3.4 is to recognize that the situation—whether Case 1, 2, or 3 is involved—is governed by the following system of *simultaneous nonlinear equations*:

Pressure drop:

$$-\Delta p = p_1 - p_2 = 2 f_\text{F} \rho u_m^2 \frac{L}{D} + \rho g \Delta z. \tag{3.47}$$

Flow rate:

$$Q = \frac{\pi D^2}{4} u_m. \tag{3.48}$$

Reynolds number:

$$\text{Re} = \frac{\rho u_m D}{\mu}. \tag{3.49}$$

Equations representing the friction-factor plot (avoid $2{,}000 < \text{Re} \leq 4{,}000$*):*

$$\begin{aligned}
\text{Re} \leq 2{,}000: &\quad f_\text{F} = \frac{16}{\text{Re}}, \\
\text{Re} > 4{,}000: &\quad f_\text{F} = \left\{-1.737 \ln \left[0.269 \frac{\varepsilon}{D} - \frac{2.185}{\text{Re}} \ln \left(0.269 \frac{\varepsilon}{D} + \frac{14.5}{\text{Re}}\right)\right]\right\}^{-2}.
\end{aligned} \tag{3.50}$$