بسم ا الر ن الرحيم

DEPARTMENT OF CIVIL ENGINEERING – KFUPM

Numerical and Statistical Methods in Civil Engineering

CE 318- 51- 2011-2012 (111)

Computer Lab. Sessions NO. 09

Subj.: Matrix solution of a system of linear equation $A\ x = b$, where matrix A is asymmetric.

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Objectives: **To determine the solution vector *x* of a general system of linear matrix equations *A x* = *b* using *Mathematica* software.**

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1. Use Mathematica software to *re-run* and *verify* the case-study provided with this sheet. As the matrix **A** is symmetric use Cholesky decomposition algorithm to determine the solution vector **x**.

2. Use Mathematica software to solve the following matrix equations using Cholesky decomposition algorithm.

Use the following matrix equation **A x = b** to complete verify your work:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 4 & 1 \\ 1 & 4 & 2 & -3 \\ 3 & 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 1 \\ -8 \end{bmatrix}$$

Verify your results using EXCEL work-sheet, and provide a print-out proof of the two out-put results.

Supporting Notes:

Using *Mathematica* to find roots of equations and to solve a system of equations.

Theorem (Cholesky Factorization). If **A** is real, symmetric and positive definite matrix, then it has a Cholesky factorization

$$A = U^T U$$

where **U** an upper triangular matrix.

Remark. Observe that $L = U^T$ is a lower triangular matrix, so that **A** = **LU**. Hence we could also write Cholesky factorization

$$A = L L^T$$

where **L** a lower triangular matrix.

Theorem (A = LU; Cholesky Factorization). Assume that **A** has a Cholesky factorization $A = U^T U$, where $L = U^T$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} u_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ u_{1,2} & u_{2,2} & 0 & 0 & \cdots & 0 \\ u_{1,3} & u_{2,3} & u_{3,3} & 0 & \cdots & 0 \\ u_{1,4} & u_{2,4} & u_{3,4} & u_{4,4} & \cdots & 0 \\ \\ u_{1,n} & u_{2,n} & u_{3,n} & u_{4,n} & \cdots & u_{n,n} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} & \cdots & u_{2,n} \\ 0 & 0 & u_{3,3} & u_{3,4} & \cdots & u_{3,n} \\ 0 & 0 & 0 & u_{4,4} & \cdots & u_{4,n} \\ \\ 0 & 0 & 0 & 0 & \cdots & u_{n,n} \end{pmatrix}$$

Or if you prefer to write the Cholesky factorization as $A = L L^T$, where $U = L^T$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & 0 & \cdots & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & l_{4,4} & \cdots & 0 \\ \\ l_{n,1} & l_{n,2} & l_{n,3} & l_{n,4} & \cdots & l_{n,n} \end{pmatrix} \begin{pmatrix} l_{1,1} & l_{2,1} & l_{3,1} & l_{4,1} & \cdots & l_{n,1} \\ 0 & l_{2,2} & l_{3,2} & l_{4,2} & \cdots & l_{n,2} \\ 0 & 0 & l_{3,3} & l_{4,3} & \cdots & l_{n,3} \\ 0 & 0 & 0 & l_{4,4} & \cdots & l_{n,4} \\ \\ 0 & 0 & 0 & 0 & \cdots & l_{n,n} \end{pmatrix}$$

The solution **X** to the linear system $AX = B$, is found in three steps:

1. Construct the matrices L and $U = L^T$, if possible.
2. Solve $LY = B$ for Y using forward substitution.
3. Solve $UX = Y$ for X using back substitution.

Example 1 (A case-study) :

Find the **A** = **LU** factorization for the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Cholesky method.

Solution 1.

Enter the matrix.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix};$$

Invoke the subroutine Cholesky.

```
Cholesky[5];
Print["L = ", MatrixForm[L]];
Print["U = ", MatrixForm[U]];
Print["The Cholesky factorization"];
Print[MatrixForm[A], " = ", MatrixForm[L], MatrixForm[U]];
```

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{7} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{7}{3}} & 0 \\ \sqrt{2} & 0 & \frac{4}{\sqrt{7}} & -2\sqrt{\frac{3}{7}} & \sqrt{2} \end{pmatrix}$$

$$U = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{7} & 0 & \frac{4}{\sqrt{7}} \\ 0 & 0 & 0 & \sqrt{\frac{7}{3}} & -2\sqrt{\frac{3}{7}} \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

The Cholesky factorization

$$\begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{7} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{7}{3}} & 0 \\ \sqrt{2} & 0 & \frac{4}{\sqrt{7}} & -2\sqrt{\frac{3}{7}} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{7} & 0 & \frac{4}{\sqrt{7}} \\ 0 & 0 & 0 & \sqrt{\frac{7}{3}} & -2\sqrt{\frac{3}{7}} \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

Verify the factorization.

```
Print["The Cholesk factorization"];
Print[MatrixForm[A], " = ", MatrixForm[L] , MatrixForm[U] ];
Print[MatrixForm[A], " = ", MatrixForm[L.U] ];
Print["Does A = L U ?"];
Print[A == L.U];
```

The Cholesk factorization

$$\begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{7} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{7}{3}} & 0 \\ \sqrt{2} & 0 & \frac{4}{\sqrt{7}} & -2\sqrt{\frac{3}{7}} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{7} & 0 & \frac{4}{\sqrt{7}} \\ 0 & 0 & 0 & \sqrt{\frac{7}{3}} & -2\sqrt{\frac{3}{7}} \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$$

Does A = L U ?

True

This time the elements on the diagonals of L and U are the same.

Remark.

The **Cholesky method** is *used only when* **A** is a *real*, *symmetric* and *positive definite* matrix.