1. Use the Newton-Raphson Method to determine the roots for the following two problems with errors in the computed roots $_{not}$ more than 0.5%.

   i) Solve part (b) of problem 6.10 [textbook page 158].

   ii) Solve problem **6.13** after re-writing (re-arranging) the nonlinear equations as

$$u(x,y) = 0.$$
$$v(x,y) = 0.$$

Solution:-

i) Required the roots of $f(x) = 8\sin(x)\, e^{-x} - 1;$

$using\ Newton\ Raphson\ Method\ (three\ iterations\ x_i = 0.3)$

Using the iteration with $x_i = 0.3$ and using the following equation; $\boldsymbol{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$

$$f'(x) = 8cos(x)e^{-x} + 8\sin(x)\, e^{-x}$$

$$\boldsymbol{x_1 = 0.3 - \frac{0.7514}{7.4132} = 0.1986}$$

$$\boldsymbol{x_2 = 0.1986 - \frac{0.294}{7.7242} = 0.1606}$$

$$\boldsymbol{x_3 = 0.1606 - \frac{0.0895}{7.815} = 0.1492}$$

$$\boldsymbol{Error = \frac{0.1606 - 0.1492}{0.1606} = 7.1}\ \%;$$

$$\boldsymbol{then\ continue\ until\ the\ error\ as\ mentioned\ above\ in\ the\ problem.}$$

ii) Required the roots of $(x-4)^2 + (y-4)^2 = 5$

$$x^2 + y^2 = 16;$$

To get the initial guess, plot the function together and the intersection points will be the initial guess. Fro graphical,x=1.8 and y=3.6.

Rearranging the equations as $u(x,y) = 5 - (x-4)^2 - (y-4)^2$

$$v(x,y) = 16 - x^2 - y^2;$$

$$\frac{\partial u}{\partial x} = -2(x-4) = 4.4; \frac{\partial u}{\partial y} = -2(y-4) = 0.8;\ \frac{\partial v}{\partial x} = -2x = -3.6; \frac{\partial v}{\partial y} = -2y = -7.2$$

The determinant of Jacobian is $4.4(-7.2) - 0.8 * -3.6 = -28.8$

The functions at the initial guess are: $u(1.8,3.6) = 0$

$$v(1.8,3.6) = -0.2$$

Using the equation 6.21 in the textbook ;

$$x = 1.8 - \frac{0 - (-0.2) * 0.8}{-28.8} = 1.80556; \quad y = 3.6 — \frac{0.2 * 4.4 - 0}{-28.8} = 3.56944$$

And continue the same way up to the number of iteration required to get the error less than 0.5%.

$\boldsymbol{x = 1.805829; y = 3.569171}$

2. Use the method of $_{Gauss\ Elimination}$ to solve textbook Problem 9.9 [textbook page 262]. Also compute the determinant of the coefficient matrix and check the accuracy of your results $\boldsymbol{x}^*$ by substitution in $\mathbf{A}\,\boldsymbol{x}^* = \boldsymbol{b}^*$ and computing the ratio of norms of vectors $\Delta\boldsymbol{b}$ and $\boldsymbol{b}$ ($_{namely}$: $\frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|}$).

From the textbook, Problem # 9.9,

$$8x_1 + 2x_2 - 2x_3 = -2; \quad 10x_1 + 2x_2 + 4x_3 = 4; \quad 12x_1 + 2x_2 + 2x_3 = 6$$

$$[A][x] = [b]$$

⇒

$$\begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 & -2 & -2 \\ 10 & 2 & 4 & 4 \\ 12 & 2 & 2 & 6 \end{bmatrix} \Rightarrow \text{switch the row three and one} \Rightarrow \begin{bmatrix} 12 & 2 & 2 & 6 \\ 10 & 2 & 4 & 4 \\ 8 & 2 & -2 & -2 \end{bmatrix}$$

Multiply row 1 by 0.8333 and then substract from row 2. Multiply row 1 by 0.6667 and substract from row 3 to get the following results:

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.33333 & 2.333 & -1 \\ 0 & 0.66667 & -3.3333 & -6 \end{bmatrix} \textit{then switch row 2 and 3} \Rightarrow$$

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.6667 & -3.3333 & -6 \\ 0 & 0.33333 & 2.3333 & -1 \end{bmatrix} \textit{then times row 2 by 0.5 and subtract from 3}$$

$$\begin{bmatrix} 12 & 2 & 2 & 6 \\ 0 & 0.6667 & -3.3333 & -6 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$ **; now from the row three we can find $x_3 = 0.5$; then back substitution to get both $x_1 \, and \; x_2$ as follows:**

$x_1 = 1.5$; $and \; x_2 = -6.5$. then check the solution by substitution in the equations.

$$\|\Delta \mathbf{b}\| = 0.0$$

3. Determine the three matrix $_{norms}$ $\|A\|_1, \|A\|_e, \|A\|_\infty$ for the matrix given in textbook Problem 10.9 [textbook page 284].

And determine the inverse of the matrix.

Solution:-

From the problem ; $[A] = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}$; *the inverse is*

0.027273	-0.00606	0.048485
0.3	0.6	0.2
-0.01818	0.115152	0.078788

We have to scale the matrix as follows: $\frac{first\ row}{10}$; $\frac{second\ row}{-9}$; $\frac{third\ row}{15}$; $\Rightarrow$

$$[A] = \begin{bmatrix} 0.8 & 0.2 & -1 \\ 1 & -0.1111 & -0.33333 \\ 1 & -0.0667 & 0.4 \end{bmatrix}$$

$\Rightarrow$

$$\|A\|_e = \sqrt{\sum_{i=1,j=1} a_{ij}{}^2} = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + ..)} = \mathbf{1.992}$$

$$\|A\|_1 = maximum\ of\ the\ summation\ of\ each\ column\ elements = \mathbf{2.8}$$

$$\|A\|_\infty = maximum\ of\ the\ summation\ of\ each\ row\ elements = \mathbf{2}$$

4. Compute the condition number for the matrix given in text book problem 10.11. Also, check if the matrix is ill-conditioned or not. Then if it is ill-conditioned specify the number of significant digits that will be lost due to ill-conditioning.

Solution:-

$$[A]=\begin{bmatrix} 0.125 & 0.25 & 0.5 & 1 \\ 0.015625 & 0.625 & 0.25 & 1 \\ 0.00463 & 0.02777 & 0.16667 & 1 \\ 0.001953 & 0.015625 & 0.125 & 1 \end{bmatrix}$$

Using the row sum,

First row = 1.875; second row=1.890625; third row =1.19907; and fourth row=1.142578

$$\|A\|_\infty = 1.890625$$

The matrix inverse is computed using excel as follows:

$$[A]^{-1}=\begin{bmatrix} 10.233 & -2.2339 & -85.3872 & 77.388 \\ -0.1008 & 1.767 & -4.3949 & 2.7283 \\ -0.6280 & -0.3716 & 30.7645 & -29.765 \\ 0.0601 & 0.0232 & -3.6101 & 4.5268 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = 175.242$$

Then the condition is computed as follows:

Cond(A)= $\|A\|_\infty * \|A^{-1}\|_\infty = 331.32$

Because of the inverse has elements more than 1, it **is ill conditioned**.

To get the digits lost by the ill condition,

Log(331.32)=2.52

So, three digits will be lost.

5. Use LU-decomposition (i.e. *Cholesky decomposition*) to solve textbook Problem 11.5 [textbook page 303].

Such that the RHS-vector of matrix equation **Ax=b** is modified to be **b**=$[100 \quad 250 \quad -50]^T$

Solution:

Q #5 :- Given Data:- • Using LU-decomposition.

$$\bullet \begin{bmatrix} 8 & 20 & 15 \\ 20 & 80 & 50 \\ 15 & 50 & 60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 250 \\ 100 \end{Bmatrix}$$

Required:- • Performing a Cholesky decomposition of the above.

Solution:-

$$L_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} l_{kj}}{l_{ii}} \quad ; \quad i = 1,2,3 \dots, k-1$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

$\Rightarrow l_{11} = 2\sqrt{2}$; $l_{21} = \frac{10}{\sqrt{2}}$; $l_{22} = \sqrt{30}$; $l_{31} = \frac{15}{2\sqrt{2}}$; $l_{32} = \frac{25}{2\sqrt{30}}$

; $l_{33} = \sqrt{\frac{80}{3}}$ $\Rightarrow [L] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ \frac{10}{\sqrt{2}} & \sqrt{30} & 0 \\ \frac{15}{2\sqrt{2}} & \frac{25}{2\sqrt{30}} & \sqrt{\frac{80}{3}} \end{bmatrix}$ $\because [L][L]^T X = b$

Note that $[L]^T X$ is D. $\Rightarrow [L][D] = b$; By solving these equations $\Rightarrow D_1 = \frac{25}{\sqrt{2}}$; $D_2 = 22.8218$; $D_3 = 8.8775$. #

$$\Rightarrow \begin{bmatrix} 2\sqrt{2} & \frac{10}{\sqrt{2}} & \frac{15}{2\sqrt{2}} \\ 0 & \sqrt{30} & \frac{25}{2\sqrt{30}} \\ 0 & 0 & \sqrt{\frac{80}{30}} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 25/\sqrt{2} \\ 22.8218 \\ 8.8775 \end{Bmatrix}$$

; solving this linear matrix equation (Back sub.)

$\Rightarrow X_3 = -1.7188$; $X_2 = 4.8828$; $X_1 = -2.7343$

6. Use *Gauss-Seidel iterative* procedure to solve textbook Problem 11.8 with an *initial* solution vector $\mathbf{c}_\circ = [30,\ 18,\ 10]$. Stop iterations when the *percent relative error* in solution vector is less than 2%.

Solution:-

Q #6. Given Data:- • Using Gauss-Seidel iterative procedure.

• Initial solution vector $C_o = [30, 18, 10]$

• stopping iterations when the percent relative error in less than 1%.

• $15C_1 - 3C_2 - C_3 = 3800$

$-3C_1 + 18C_2 - 6C_3 = 1200$

$-4C_1 - C_2 + 12C_3 = 2350$

Required:- • Solution of the above problem.

Solution:- To solve this problem; $C_1 = \frac{\text{first equation} - 15C_1}{\text{coefficient of } C_1}$

$C_2 = \frac{\text{second equation} - 18C_2}{\text{coefficient of } C_2}$; $C_3 = \frac{\text{third equation} - 12C_3}{\text{coefficient of } C_3}$

Using the initial values $C_o = [30, 18, 10]$

⇒ First iteration:- $C_1 = 257.6$; $C_2 = 112.935$; $C_3 = 291.411$

Use the resulting values from the first iteration into the second iteration

Second iteration ; $C_1 = 295.33$; $C_2 = 212.924$; $C_3 = 312.0195$

The same repeatation is followed until the relative error in less than 1% where the relative error $\varepsilon_r = \frac{\text{New value} - \text{old value}}{\text{New value}}$;

after almost five iterations ⇒

$C_1 = 320.058$; $C_2 = 227.067$

$C_3 = 321.442$ #