1. Determine the parameters *a* to *h* [used below: to define *S*(*x*)] so that the function *S* (*x*) defines a *natural* cubic spline:

$$S(x) = ax^3 + bx^2 + cx + d \qquad x \in [-1,0]$$

$$S(x) = ex^3 + fx^2 + gx + h \qquad x \in [0,\ 1]$$

such that the following conditions are satisfied:

S (-1) = 1; S (0) = 2; S (1) = -1.

Hint-2: *Recall* the required conditions on *slopes* at internal node, and on *curvatures* of the splines at end nodes.

Solution:-

Using the given conditions; S (-1) = 1 ⇒

-a + b –c + d = 1 --------(3) ; S (0) = 2 ⇒ d = 2 , from Eq. (1), and h = 2 from Eq. (2).

For Eq. (2), S (1) = -1 ⇒ e + f + g + 2 = -1 ⇒

⇒ e + f + g = -3 ----------- (4) ; from Eq. (3)

⇒ -a + b –c = -1 -----------(5)

Hence $f_{i-1}'(x) = f_i'(x)$; ⇒

$$s'(x) = 3ax^2 + 2bx + c \qquad x \in [-1,0] - - - -(6)$$

$$s''(x) = 6ax + 2b \qquad x \in [-1,0] - - - -(7)$$

$$s'(x) = 3ex^2 + 2fx + g \qquad x \in [0,1] - - - -(8)$$

$$s''(x) = 6ax + 2f \qquad x \in [0,1] - - - -(9)$$

$hence\ f''(x_0) = f''(x_n)$ $\Rightarrow s''(-1) = s''(1) = 0.0$ by using the equation above, a = 1/3 b & e = -1/3 f

& hence, $f_{i-1}'(x) = f_i'(x) \Rightarrow s'(0) Eq.(6) = s'(0) Eq.(8) \quad \Rightarrow c = g$

$$s''(0) Eq.(7) = s''(0) Eq.(9) \quad \Rightarrow f = b$$

$$\Rightarrow by\ subistitution\ into\ Eq's 4\ and\ 5 \ \Rightarrow$$

$$a = -\frac{2}{3}\ ; b = -2\ ;\ c = \frac{1}{3}\ ; d = 2\ ; e = \frac{2}{3}\ ; f = -2; g = \frac{1}{3}\ and\ h = 2$$

OK

2. Use the *Trapezoidal* rule with n = 2 and 4 to evaluate the following integral (with x evaluated is in *radians*).

$$I = \int_0^{\pi/2} (2 + \sin 4x)dx$$

Compare the *true percent* errors; then use *Richardson's* extrapolation to obtain an improved value of the integral evaluated in problem 1 above and re-compute the *percent* error.

Solution:-

$$When\, n = 2\, ;\; x_0 = 0\, , x_n = \frac{\pi}{2}, \qquad x_1 = \frac{\pi}{4} \;\; ; \;\; f(0) = 2\, , f\left(\frac{\pi}{4}\right) = 2\, , f\left(\frac{\pi}{2}\right) = 2$$

$$I = (b-a)\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n} = \left(\frac{\pi}{2} - 0\right)\frac{2 + 2(2) + 2}{2 * 2} = 3.142$$

The error can be found using the exact value as follows:

$$I = 2x\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}cos4x\Big|_0^{\frac{\pi}{2}} = \pi - 0 = 3.142 \quad \Rightarrow$$

The error as percent is $\varepsilon_t = \frac{3.142 - 3.142}{3.142} * 100 = 0\ \%$

$$When\, n = 4\, ;\; x_0 = 0\, , x_n = \frac{\pi}{2}, x_1 = \frac{\pi}{8} \;\; ; \; x_2 = \frac{\pi}{4}\; ; x_3 = \frac{3\pi}{8}\; ;$$

$$f(0) = 2\, , f\left(\frac{\pi}{8}\right) = 3; f\left(\frac{\pi}{4}\right) = 2\, ,; f\left(\frac{3\pi}{8}\right) = 1; f\left(\frac{\pi}{2}\right) = 2$$

$$I = \left(\frac{\pi}{2} - 0\right)\frac{2 + 2(3 + 2 + 1) + 2}{2 * 4} = 3.1416$$

$$\Rightarrow$$

The error as percent is $\varepsilon_t = \frac{3.142 - 3.142}{3.142} * 100 = 0\ \%$

Using Richardson's Extrapolation, the integration can be evaluated as follows:-

$$\boldsymbol{I = \frac{4}{3}\, I(h_2) - \frac{1}{3} I(h_1)}$$

$$\boldsymbol{I(h_1) = \pi \;\; ; and\; I(h_2) = \pi \;\;\; ; \Rightarrow I = \pi}$$

$$\varepsilon_t = \frac{3.142 - 3.142}{3.142} * 100 = 0\ \% \qquad\qquad \textbf{OK}$$

3. Use *Gauss-Legendre* formulas to evaluate the integral given in problem 22.8 [text-book page 631] with *two*-, and *four*-point formulas.

The problem is evaluating the following:

$$I = \int_{-3}^{3} \frac{dx}{1+x^2}$$

Solution:-

a. Two point Gauss-Legendre Formula

$$I = c_0 f(x_0) + c_1 f(x_1)$$

$$c_0 = 1;\ c_1 = 1.0 \quad but\ x_0 = \frac{(b+a)+(b-a)x_{0(from\ Gauss)}}{2} \quad ; \quad and$$

$$x_1 = \frac{(b+a)+(b-a)x_{1(from\ Gauss)}}{2} \quad ; \quad dx = \frac{b-a}{2} = \frac{6}{2} = 3$$

$$\Rightarrow x_0 = \frac{(3-3)+(3+3)(-0.577350269)}{2} = -1.732051$$

$$f(x_0) = \frac{1}{1+(-1.732051)^2} * 3 = 0.75$$

$$x_1 = 3 * 0.577350269 = 1.732051$$

$$f(x_{01}) = \frac{1}{1+(1.732051)^2} * 3 = 0.75$$

$$\boldsymbol{I = 0.75 + 0.75 = 1.5}$$

The same procedure will be followed in case of four Gauss Legendre formulas to get

$$I = 2.1898$$

OK

4. Solve problem 5.5 [textbook: page 131] to determine the *real* root of the function within the range of values of *x* between 0.4 and 1.2.

The Problem is as follows:

$Sin(x) = x^3$, *Perform the computation until* ε_a *is less than* $\varepsilon_s = 2\%$.

Solution:-

Using the bisection method for the solution as follows:-

$$X_r = \frac{0.4 + 1.2}{2} = 0.8 \;\; ;$$

$$Check \;\; f(0.4) * f(0.8) \quad which\ equals\ to \;\; 0.0668$$

$$then\ recalculate X_r = \frac{0.8 + 1.2}{2} = 1.0 \;\; ;$$

$$Clculate\ the\ error\ \varepsilon_d = \left|\frac{1 - 0.8}{1}\right| * 100 = 20\%$$

$$repeat\ the\ previous\ steps\ until\ the\ \ \varepsilon_d\ less\ than\ 2\%.$$

The final step is

$$then\ recalculate X_r = \frac{0.906 + 0.938}{2} = 0.922 \;\; ;$$

$$Clculate\ the\ error\ \varepsilon_d = \left|\frac{0.922 - 0.938}{0.922}\right| * 100 = 1.7\%$$

⇒ The real root of the function is x =0.922