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RBF-based meshless methods for 2D elastostatic problems

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Abstract

The work presented here concerns the use of radial basis functions (RBFs) for the analysis of two dimensional elastostatic problems. The basic characteristic of the formulation is the definition of a global approximation for the variables of interest in each problem (the deflection for the plate bending problem and the stress function for the stretching plates) from a set of RBFs conveniently placed (but not necessarily in a regular manner) at the boundary and in the domain. Depending on the type of collocation chosen, non-symmetric or symmetric systems of linear equations are obtained. Comparisons are made with other results available in the literature.

Keywords: Collocation techniques; Hermite collocation; Meshless; Radial basis functions

1. Introduction

Traditional numerical approaches for the approximate solution of partial differential equations (PDEs) require the use of a mesh: a domain mesh in the case of domain methods such as the finite difference methods, the finite element methods and the finite volume methods, or a boundary mesh (and sometimes, for nonlinear problems, a domain one as well) for boundary methods such as the boundary element method.

The main difficulties associated to the use of a mesh are the definition of the mesh itself. In fact, a good amount of time has been spent by researchers on just trying to develop specific procedures to define the mesh or to properly refine it.

It has been a goal of many throughout the computational mechanics community to be able to obtain a good numerical solution by means of a meshless technique, that is, without the need for a mesh or with just a minimal mesh.

Bearing this goal in mind, there have been several directions of research which can, basically, be included in two families: methods based on global approximations and methods based on local approximations.

The work of Lucy [17] on smoothed particle hydrodynamics (SPH) may be seen as the precursor of most local approximation meshless techniques. This technique, a development on particle methods, is based on an integral form of approximation whereby the integrand includes a weight function of Dirac character; this concept is known as 'reproducing kernel' as in the work of Liu and co-workers [15,16]. In these methods, the weight function task is that of including, of bringing into the approximation, the effect of the neighbouring particles.

In the context of finite elements, a similar idea was used by Nayroles et al. [18] in what is known as the Diffuse Element Method; nodal, not elemental, 'shape' functions are built which take into consideration the effect of, again, neighbouring nodes according to a 'weight function', that is, in a diffuse manner.

Further work on the concepts of diffuse nodal shape functions was then carried out by Belytschko and co-workers in what is known as the Element-Free Galerkin method [3]. These authors recognized that diffuse shape functions are just variants of a more general approach known as moving least squares, of Lancaster and Salkauskas [10].

The concept of partition of unity (which, in a way, was being used in the previous techniques) was then formally introduced into the family of meshless techniques independently by Duarte and Oden [5] (the h–p clouds method) and Babuska and Melenk [2], thus allowing for an easier (than with the EFGM) nodal enrichment of the approximations.

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None of the above referred methods is entirely meshless: integrations, on a background cell which resemble elements, have to be carried out.

Atluri and co-workers [1] have presented techniques, the meshless local boundary equation method and the meshless local Petrov–Galerkin approach, whereby local forms are defined on simpler subdomains (tipically circles or spheres) thus avoiding the background cells.

More recently, Liu and co-workers [13,14,22], proposed using local interpolations based on radial basis functions (RBFs) (instead of based on moving least squares) in what, given the fact that RBFs are normally used in a global context, somehow links local and global approximations.

Global approximations rely, as the name indicates, on the use of approximating functions that exist and are valid throughout the domain. Two main meshless approaches can be referred here: Trefftz based methods [20] which are inherently meshless especially in the indirect collocation variant, [9,11], and the approach followed in this work which is based on the use of RBFs and started from the work of Hardy [7] and, in what concerns application to PDEs, Kansa [8].

The work herein concerns further developments of an RBF-based meshless method previously applied, by the author, to Kirchhoff plate bending problems [12]. These improvements consist in the appropriate consideration of irregular distribution of collocation points and, also, in the extension of the formulation to the analysis of plane states by means of stress functions.

2. Radial basis functions and interpolation

A RBF is a localized function, localized about a center x_j, that depends only (apart from some parameters) upon the distance $r = \|x - x_j\|$ between the center and a generic point x, that is, $\phi(r)$.

RBFs have been initially used for scattered data fitting and general multi-dimensional data interpolation problems, see Ref. [7], and were later applied by Kansa [8] for the analysis of PDE. The different families of RBF available may be divided into two categories: globally and compactly supported ones.

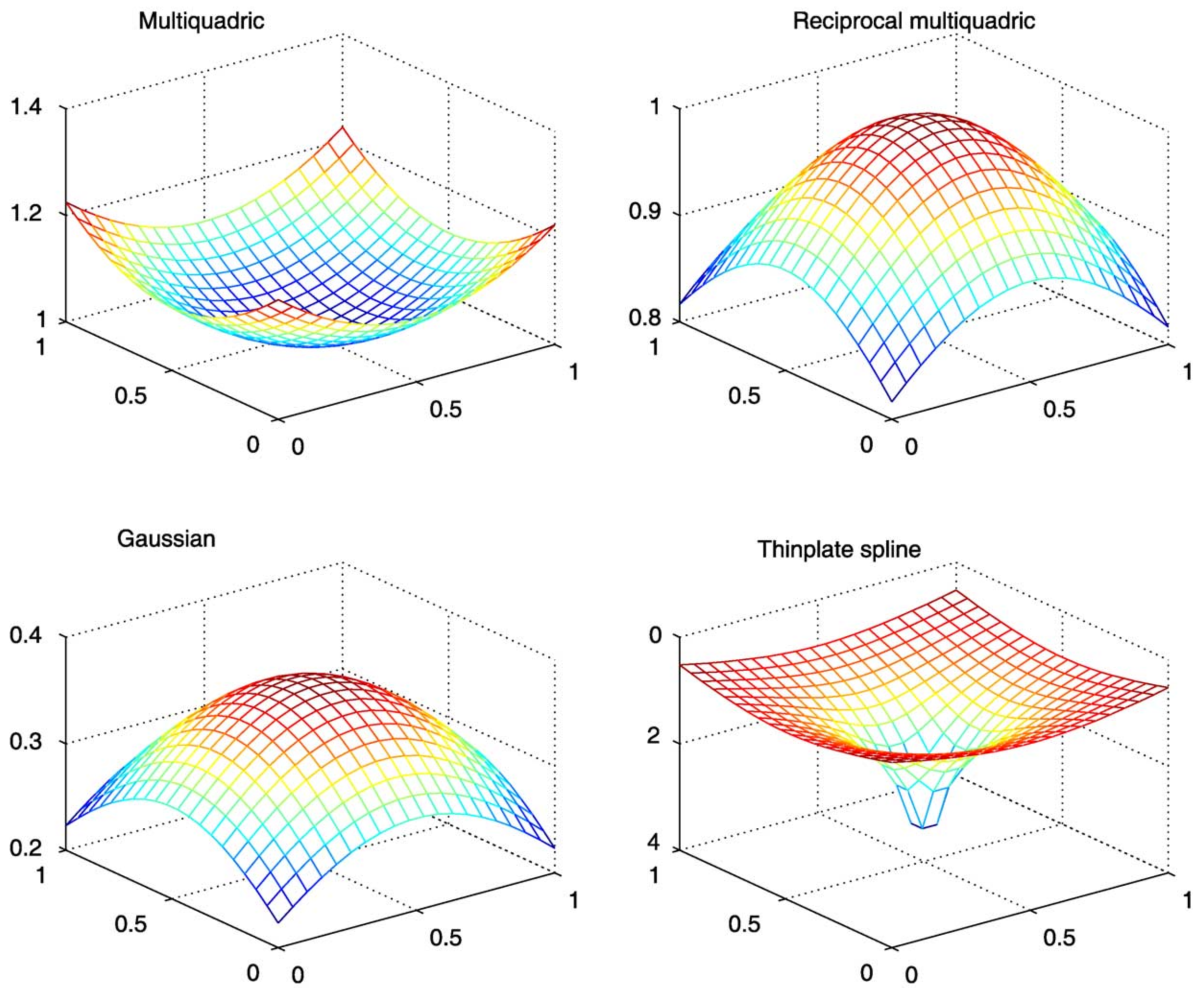

Fig. 1. Globally supported RBFs.

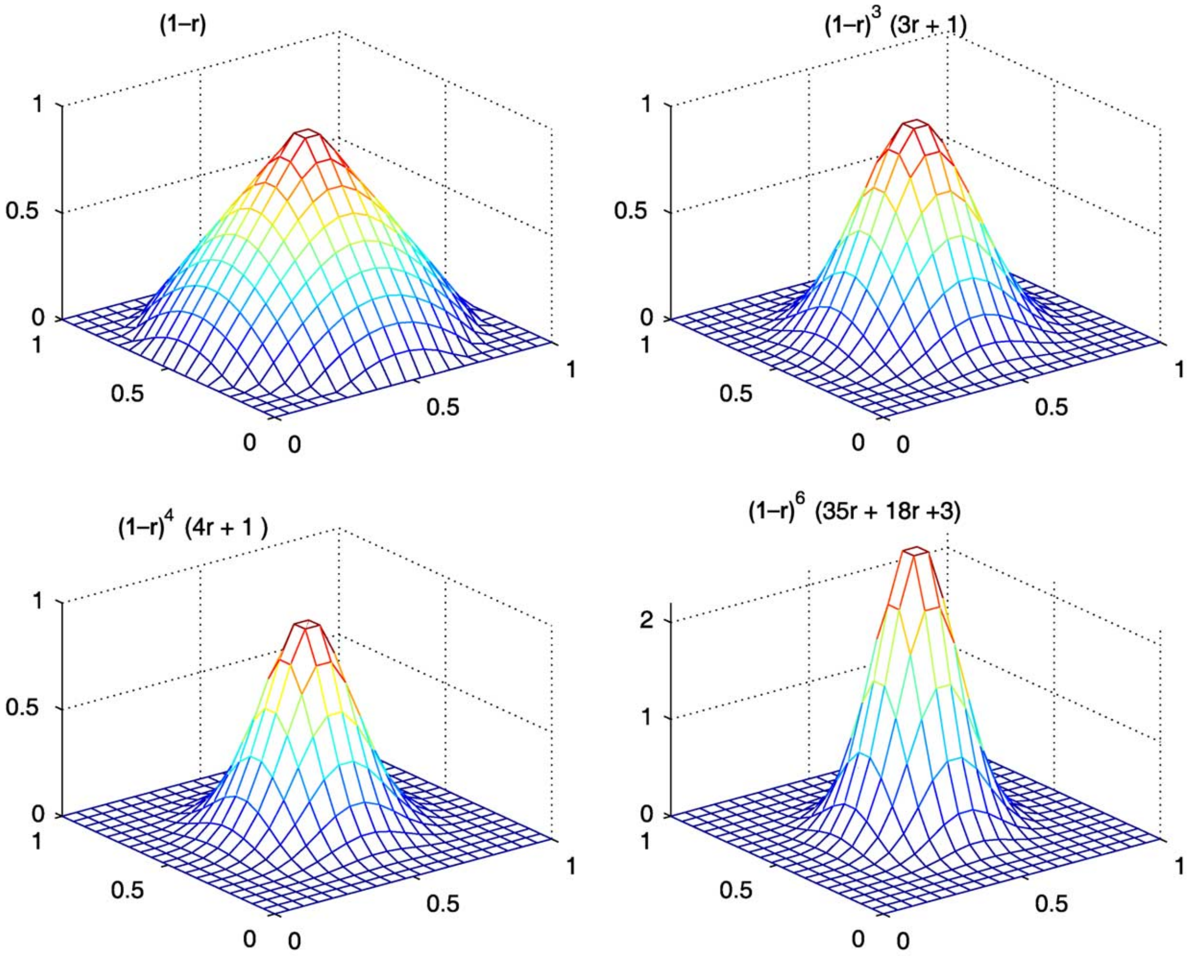

Fig. 2. Compactly supported RBFs.

In Fig. 1 the following globally supported RBFs are represented (for a fixed set of parameters):

Multiquadrics $\sqrt{(x-x_j)^2+c_j^2}$

Gaussians $\exp\left(-\frac{r^2}{\sigma^2}\right)$

Thin plate splines $r^2\beta \ln r$

In Fig. 2 examples of the following compactly supported RBFs are represented:

- Wu and Wendland, $(1-r)_+^n p(r)$ where $p(r)$ is a polynomial and $(1-r)_+^n$ is 0 for r larger than the support;
- Buhmann, $(1/3)+r^2-(4/3)r^3+2r^2\ln r$.

The work presented here focuses on the use of globally supported RBFs. Research is being carried out on the use of the compactly supported RBFS.

Be it with globally or compactly supported functions the starting point to define an interpolation to a given function (or a set of N scattered data points, $f(x_i)$) with RBFs is the following expression:

$$s(x)=\sum_{j=1}^{N}\alpha_j\phi(\|x-x_j\|) \tag{1}$$

where x_j are the coordinates of the centers of the RBFs $\phi(\|x-x_j\|)$.

The α_j unknowns are obtained by setting up an appropriate system of N linear equations of the type:

$$s(x_i)=f(x_i)=\sum_{j=1}^{N}\alpha_j\phi(\|x_i-x_j\|) \tag{2}$$

where x_i are the points on which interpolation data $f(x_i)$ is known.

In order to ensure the uniqueness of the interpolation, see Ref. [7], a constant, α_{N+1}, is added to the expansion and a constraint on the coefficients is imposed in the form $\sum_{j=1}^{N}\times\alpha_j=0$. This results in another equation being added to the system, that is, all entries at the $N+1$th row and column take the value one except at the diagonal where it is zero and the right-hand side is also zero.

An alternative form to obtain an interpolant was described by Fasshauer [6] and leads to inherently symmetric and non-singular systems of linear equations.

Assume a set of points x_j, $j = 1, \ldots, N$, a linearly independent set of continuous linear functionals, $\mathbf{L}^T = [L_1, \ldots, L_N]$ and a set of values $L_j f$ constituting the right-hand side vector $\mathbf{F}^T = [F_1, \ldots, F_N]$. The goal is to find an interpolant of the form:

$$s(x) = \sum_{k=1}^{N} \alpha_k L_k^{\varepsilon} \phi(\|x - \varepsilon\|), \tag{3}$$

where x is a generic point and ε represents a set of centers for the RBFs by satisfying

$$L_j s = L_j f, \qquad j = 1, \ldots, N \tag{4}$$

at all points x_j, $j = 1 \ldots N$ where $L_k g(x) := (Lg(x))|_{x=x_k}$ and $L_k \varepsilon g(\|x - \varepsilon\|)$ is the function of x, obtained when $\mathbf{L}$ acts on $g(\|x - \varepsilon\|)$ as a function of ε and then evaluated at $\varepsilon = \varepsilon_k$.

The unknowns α_k in expansion (3) are obtained by satisfying Eq. (4) and this leads to the system of equations:

$$\mathbf{A}\alpha = \mathbf{F} \tag{5}$$

where the elements of matrix $\mathbf{A}$ are of the type:

$$A_{jk} = L_j^x L_k^{\varepsilon} g(\|x - \varepsilon\|), \qquad j, k = 1, \ldots, N \tag{6}$$

and $L_j^x g(\|x - \varepsilon\|)$ is the function of ε, obtained when L acts on $g(\|x - \varepsilon\|)$ as a function of x and then evaluated at $x = x_j$.

The question that may arise now is why use RBFs to build approximations? One possible answer is that traditional interpolation methods, such as splines, may be of difficult application or may lead to poor approximations on irregular grids.

As for RBFs, although the quality of the approximation may vary with the position and number of RBFs centers it is usually of reasonable quality even for an irregular distribution of centers. This shows that, in effect, approximations using RBFs are truly meshless.

3. Application of the RBF interpolation concept to the solution of partial differential equations

The extension of both interpolation techniques described above to the analysis of PDEs arising in computational mechanics was presented by Kansa [8] (for the asymmetric form and is therefore known as the asymmetric collocation method or Kansa's approach) and by Fasshauer [6] and Wu [24] (for the symmetric form or Hermite collocation approach).

Kansa considered an approximation $u_h(X)$ to the PDE in the form:

$$u_h(x) = \sum_{j=1}^{N} \alpha_j \phi(\|x - x_j\|) \tag{7}$$

where x_j, f_j, $j = 1, \ldots, N$, define a data set.

The PDE (an elliptic one, for example) consists in an interior $\mathscr{L}_{\mathscr{I}}$ and a boundary $\mathscr{L}_{\mathscr{B}}$ operators:

$$\mathscr{L}u = \mathscr{F} \tag{8}$$

where $\mathscr{L}^{\mathrm{T}} = [\mathscr{L}_{\mathscr{I}} \quad \mathscr{L}_{\mathscr{B}}]$ and $\mathscr{F}^{\mathrm{T}} = [\mathscr{F}_{\mathscr{I}} \quad \mathscr{F}_{\mathscr{B}}]$ is the right-hand side vector.

The unknown coefficients α_j are determined by solving the system of N linear equations formed by applying (that is, by collocating) the operators $\mathscr{L}_{\mathscr{I}}$ and $\mathscr{L}_{\mathscr{B}}$ to the approximation (7) at N selected points or, better, $\mathscr{L}_{\mathscr{I}}$ is applied to $N - M$ points and $\mathscr{L}_{\mathscr{B}}$ is applied to the remaining M points (or boundary collocation points.

$$\mathscr{L}_{\mathscr{I}} u_h(x_i) = \sum_{j=1}^{N-M} \alpha_j \mathscr{L}_{\mathscr{I}} \phi(\|x_i - x_j\|) \tag{9}$$

$$\mathscr{L}_{\mathscr{I}} u_h(x_i) = \sum_{j=N-M+1}^{N} \alpha_j \mathscr{L}_{\mathscr{B}} \phi(\|x_i - x_j\|) \tag{10}$$

Fasshauer considered an approximation $u_h(X)$ to the PDE in the form:

$$u_h(x) = \sum_{k=1}^{N-M} \alpha_k \mathscr{L}_{\mathscr{I}_k}^{\varepsilon} \phi(\|x - \varepsilon_k\|) + \sum_{k=N-M+1}^{N} \alpha_k \mathscr{L}_{\mathscr{B}_k}^{\varepsilon} \phi(\|x - \varepsilon_k\|) \tag{11}$$

The unknown coefficients α_j are determined by solving the system of N linear equations formed by applying the operators $\mathscr{L}$ to the approximation (3) at N selected points:

$$\mathscr{L}_{\mathscr{I}_j}^{x} u_h(x_j) = \sum_{k=1}^{N-M} \alpha_k \mathscr{L}_{\mathscr{I}_j} x \mathscr{L}_{\mathscr{I}_k}^{\varepsilon} \phi(\|x_j - \varepsilon_k\|) + \sum_{k=N-M+1}^{N} \times \alpha_k \mathscr{L}_{\mathscr{I}_j} x \mathscr{L}_{\mathscr{B}_k}^{\varepsilon} \phi(\|x_j - \varepsilon_k\|) \tag{12}$$

for the interior collocation points and,

$$\mathscr{L}_{\mathscr{B}_j}^{x} u_h(x_j) = \sum_{k=1}^{N-M} \alpha_k \mathscr{L}_{\mathscr{B}_j} x \mathscr{L}_{\mathscr{I}_k}^{\varepsilon} \phi(\|x_j - \varepsilon_k\|) + \sum_{k=N-M+1}^{N} \times \alpha_k \mathscr{L}_{\mathscr{B}_j} x \mathscr{L}_{\mathscr{B}_k}^{\varepsilon} \phi(\|x_j - \varepsilon_k\|) \tag{13}$$

for the boundary collocation points.

4. Formulation of plate bending and plate stretching problems

The application of the concepts described above is now illustrated for two types of elastostatic problems, namely, the cases of bending and stretching of plates. Details of the application to the first of these cases, plate bending, may be found in Ref. [12]. In the work presented here the formulation is further developed to allow for irregular distribution of RBF centers and collocation points.

The starting point, in the definition of the approximate solutions, requires the appropriate setting up of the field and boundary equations of the problem, that is, the governing equations.

Consider first the case of plate bending. The differential equation of the deflection surface of an homogeneous, isotropic, arbitrary thin plate under bending is the well known Lagrange equation:

$$\nabla^4 w = \frac{p}{D} \tag{14}$$

subjected, at every boundary point, to two out of the set of the following four boundary conditions: $\bar{w}$, $(\partial \bar{w})/(\partial n)$, $\bar{M}_n$ and $\bar{V}_n$, representing, respectively, the value of the deflection itself, the normal derivative of the deflection (the normal slope), the normal bending moment and the normal effective shear.

The above expressions may be written, in terms of the deflection, in the following form:

$$\mathscr{L} w = \mathscr{F} \tag{15}$$

or, in expanded form:

$$\mathscr{L} = \begin{vmatrix} L_{\nabla^4} \\ L_w \\ L_{\theta_n} \\ L_{M_n} \\ L_{V_n} \end{vmatrix} = \begin{vmatrix} \nabla^4 \\ I \\ \dfrac{\partial}{\partial x}\cos\alpha + \dfrac{\partial}{\partial y}\sin\alpha \\ -D\left\{\nu\nabla^2 + (1-\nu)\left(\cos^2\alpha\dfrac{\partial^2}{\partial x^2} + \sin^2\alpha\dfrac{\partial^2}{\partial y^2} + \sin^2\alpha\dfrac{\partial^2}{\partial x\,\partial y}\right)\right\} \\ -D\left\{\cos\alpha\dfrac{\partial}{\partial x}\nabla^2 + \sin\alpha\dfrac{\partial}{\partial y}\nabla^2 + (1-\nu)\dfrac{\partial}{\partial t}\left[\cos 2\alpha\dfrac{\partial^2}{\partial x\,\partial y} + \dfrac{1}{2}\sin 2\alpha\left(\dfrac{\partial^2}{\partial y^2} - \dfrac{\partial^2}{\partial x^2}\right)\right]\right\} \end{vmatrix}$$

and

$$\mathscr{F} = \begin{vmatrix} F_{\nabla^4} \\ F_w \\ F_{\theta_n} \\ F_{M_n} \\ F_{V_n} \end{vmatrix} = \begin{vmatrix} \dfrac{p}{D} \\ \bar{w} \\ \dfrac{\partial \bar{w}}{\partial n} \\ \bar{M}_n \\ \bar{V}_n \end{vmatrix}$$

Now, considering only the Hermite collocation approach, the approximation takes the form:

$$\begin{aligned} w_h(\mathbf{x}) = & \sum_{k=1}^{N_{\nabla^4}} \alpha_k L^{\varepsilon}_{\nabla^4,k}\phi(\|\mathbf{x}-\varepsilon_k\|) + \sum_{k=N_{\nabla^4}+1}^{N_{\nabla^4}+N_w} \alpha_k L^{\varepsilon}_{w,k}\phi(\|\mathbf{x}-\varepsilon_k\|) \\ & + \sum_{k=N_{\nabla^4}+N_w+1}^{N_{\nabla^4}+N_w+N_{\theta_n}} \alpha_k L^{\varepsilon}_{\theta_n,k}\phi(\|\mathbf{x}-\varepsilon_k\|) \\ & + \sum_{k=N_{\nabla^4}+N_w+N_{\theta_n}+1}^{N_{\nabla^4}+N_w+N_{\theta_n}+N_{M_n}} \alpha_k L^{\varepsilon}_{M_n,k}\phi(\|\mathbf{x}-\varepsilon_k\|) \\ & + \sum_{k=N_{\nabla^4}+N_w+N_{\theta_n}+N_{M_n}+1}^{N_{\nabla^4}+N_w+N_{\theta_n}+N_{M_n}+N_{V_n}} \alpha_k L^{\varepsilon}_{V_n,k}\phi(\|\mathbf{x}-\varepsilon_k\|) \end{aligned} \tag{16}$$

The deflection approximation is known when all the α_j coefficients are determined and this is done exactly in the way described in the preceding section thus leading to a symmetric system of linear equations $\mathscr{L}w = \mathscr{F}$ by collocating (that is, by applying the appropriate differential operator) at each interior or boundary collocation point. Details may be found in Leitáo [12].

The application to plane problems may be easily built on the recognition of the analogy between the stress functions in plane problems and the deflections of thin plates under bending, see Fraeijs de Veubeke [21].

Regarding this application reference should be made to the work of Zhang et al. [25] that have recently presented an application of RBFs to plane states but considering the displacement form of the governing equation (Navier equation) instead of stress functions. Another difference of their work in relation to the present work is that their Hermite collocation follows a slightly different approach which does not lead to a symmetric system of linear equations.

Following the above referred analogy between the stress functions in plane problems and the deflections of thin plates under bending, plate stretching problems may also be represented by the biharmonic equation when expressed in terms of stress functions Φ:

$$\nabla^4 \phi = 0 \tag{17}$$

and assuming constant temperature.

The stress function is defined from:

$$\begin{aligned} \sigma_x &= \frac{\partial^2 \Phi}{\partial y^2} + f \\ \sigma_y &= \frac{\partial^2 \Phi}{\partial x^2} + f \\ \tau_{xy} &= \frac{-\partial^2 \Phi}{\partial x\,\partial y} \end{aligned} \tag{18}$$

in which f is a body-force potential.

The definition of the boundary conditions is, although slightly more difficult, similar to those for plate bending in the sense that they should be defined in the normal and tangential directions. For the sake of simplicity only

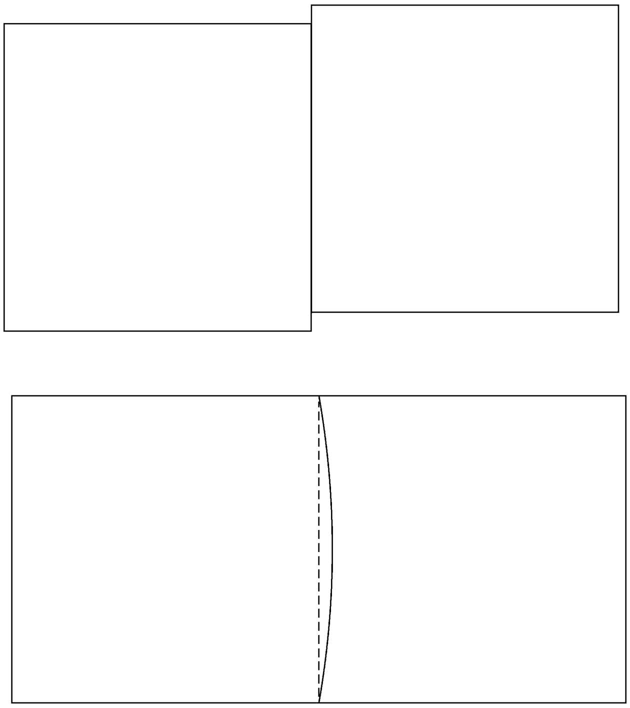

Fig. 3. Defects in compatibility.

boundaries with constant curvature (s coordinate coincides with t coordinate) will be considered here.

The boundary conditions, expressed in terms of the stress function, are as follows:

$$\frac{\partial^2 \Phi}{\partial t^2} + f = \bar{\sigma}_n$$

$$-\frac{\partial^2 \Phi}{\partial t\,\partial n} = \bar{\sigma}_{nt}$$

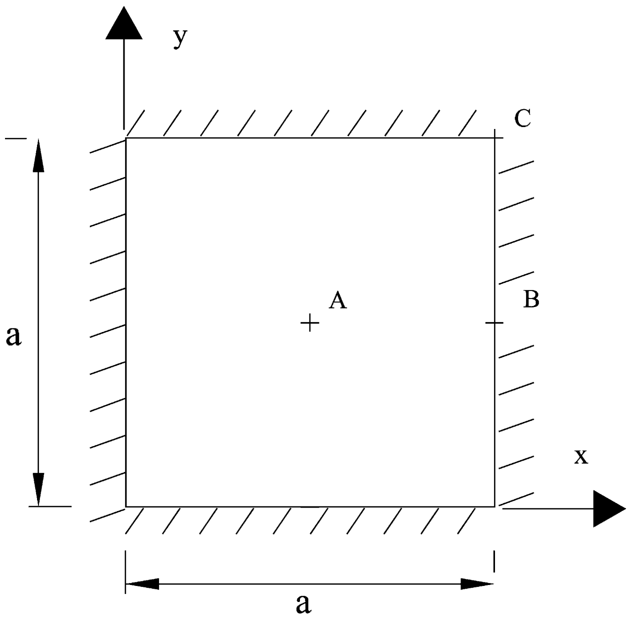

Fig. 4. Clamped square plate subjected to a uniform load.

Table 1
Clamped square plate under uniform load

Grid	Center of plate (A)		Midside point (B)	
	$100w^{adim}$	$10M_{xx}^{adim}$	V_n^{adim}	$10M_n$adim
Regular	0.1265	0.2291	−0.4412	−0.5135
$k=2$	0.1265	0.2291	−0.4429	−0.5135
$k=1.5$	0.1264	0.2292	−0.4356	−0.5082
$k=1$	0.1256	0.2278	−0.4423	−0.5096

Effect of irregular distribution of points on normalized displacement, bending moment and shear at the center of plate (point A) and middle of side (point B), Fig. 4.

$$\epsilon_t = \frac{\partial \bar{u}_t}{\partial s}$$

$$2\frac{\partial \epsilon_{nt}}{\partial s} - \frac{\partial \epsilon_t}{\partial n} = \frac{\partial^2 \bar{u}_n}{\partial s^2}$$

If the static type of boundary conditions are somewhat obvious that is not the case for the compatibility/kinematic type of boundary conditions.

The first of the kinematic boundary conditions expresses the compatibility along the tangential direction whereas the second one expresses the compatibility along the normal direction. This is illustrated in the Fig. 3 where the loss of tangential contact may represent a compatibility defect that should be restrained (in case the boundary condition is such that tangential displacements are not allowed) and where the loss of normal contact may represent a compatibility defect that should be restrained (in case the boundary condition in that direction is such that normal displacements are not allowed).

The same form of approximation used for plate bending is used now for plane problems. The stress function approximation is known when all the α_j coefficients are determined. In order to achieve that, a system of linear equations $\mathscr{L}w = \mathscr{F}$ has to be defined by collocating (that is,

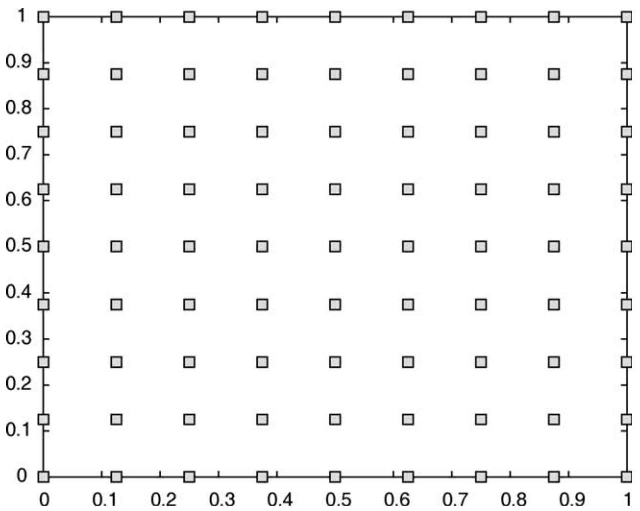

Fig. 5. Regular distribution of points, $k=\infty$.

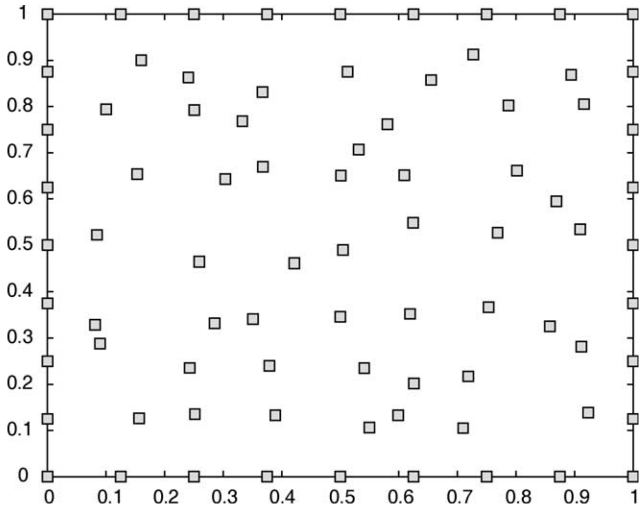

Fig. 6. Irregular distribution of points, $k = 2$.

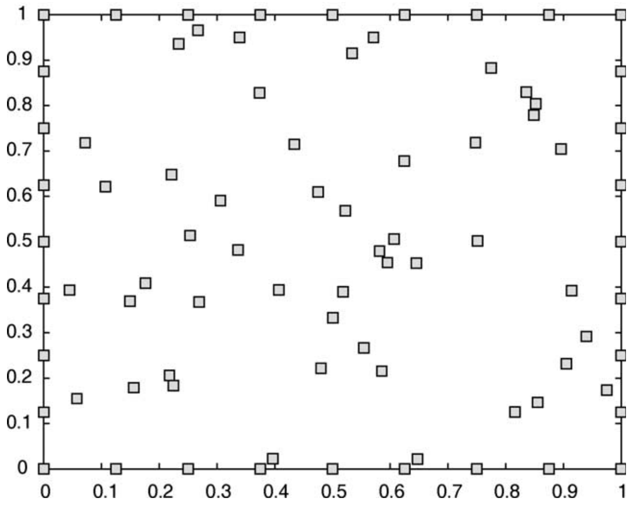

Fig. 8. Irregular distribution of points, $k = 1$.

by applying the appropriate differential operator) at each interior or boundary collocation point.

5. Numerical implementation

Techniques based on collocation are relatively easy to code. In the particular case of the Hermite collocation approach described above the main difficulty is the need to actually code all the combined operators required to create each entry $A_{jk} = L^{x}_{\theta_n} L^{\varepsilon}_{V_n} \phi(\|x_j - \varepsilon_k\|)$ in the system matrix **A**. By using a symbolic mathematics software, Mathematica [23], and Matlab [19] this task is made quite simple.

One of the key aspects affecting the quality of the approximation is the number and position of the collocation/RBF center points. Results are particularly affected by the enforcement of boundary conditions. Variations on the position and number of domain points have a somewhat lower effect. As a matter of fact it will be shown that a highly irregular, nearly random, distribution of points is able to model the problem reasonably well. Irregular grids with a certain degree of randomness were obtained (starting from regular grids) by adding a random variation of each internal point coordinates in both directions.

Different schemes exist for obtaining the random points. In this case, the rand() function of Matlab was used to obtain a random variation on a regular distribution of points; a random increment obtained by dividing the rand() by a factor of the regular grid spacing is added to each point coordinates. The factor of the regular grid spacing is, in the examples shown later, defined as the parameter k; the bigger the k the smaller is the variation over the coordinates.

As referred earlier only the multiquadric RBF was used in this work. This family of RBFs requires the definition of a shape parameter, c, that may have a significant effect on the approximation and, therefore, on the quality of the results.

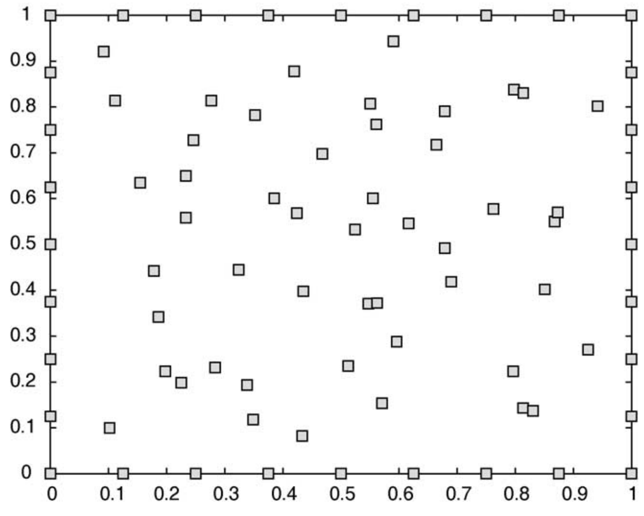

Fig. 7. Irregular distribution of points, $k = 1.5$.

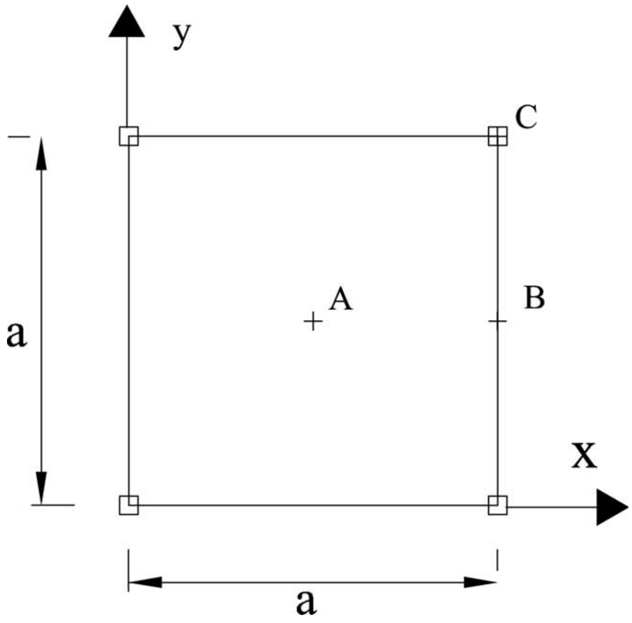

Fig. 9. Plate supported at the corners.

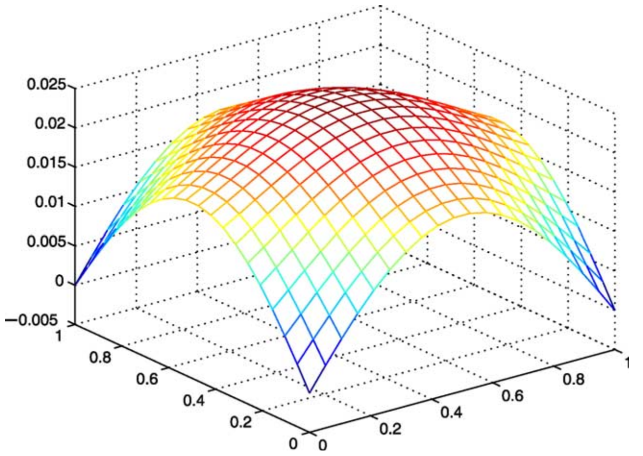

Fig. 10. Deformed shape for plate supported at corners.

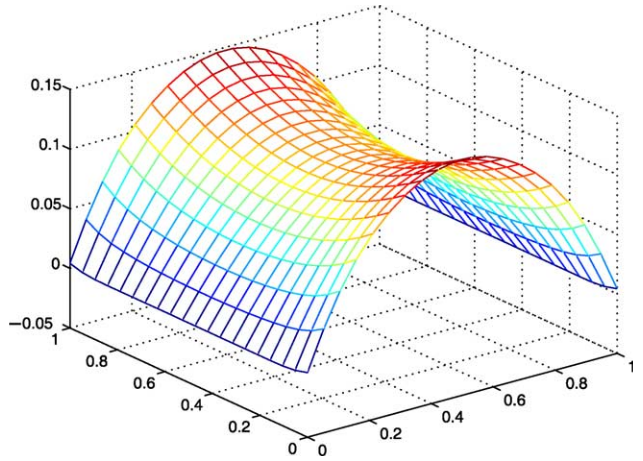

Fig. 12. M_x moment for the plate supported at corners.

In this work, emphasis was on the sensitivity to irregularity in the distribution of points and not on sensitivity to the shape parameter: a unit shape parameter was used throughout.

Concerning the effect of c on convergence an interesting paper was submitted recently by Cheng et al. [4] where it is stated that errors in the approximations tend to zero as the shape parameter goes to infinity (as the flatness of the radial function increases the error decreases). Of course this has to be balanced with the machine precision, that is, as c increases so the condition number and if this gets too large then results are only acceptable up to the point where the precision of the machine is respected.

The convergence property referred above, may diminish, in a certain way, two of the main drawbacks of global approximation approaches: fully populated system matrices and difficulties with large number of unknowns. It is then possible, within machine precision, to obtain better approximations by increasing the shape parameter instead of increasing the number of unknowns (or centers of RBF). Complimentary, for larger applications and in the context of global approximations, it is always recommendable to decompose the domain into a series of simpler ones.

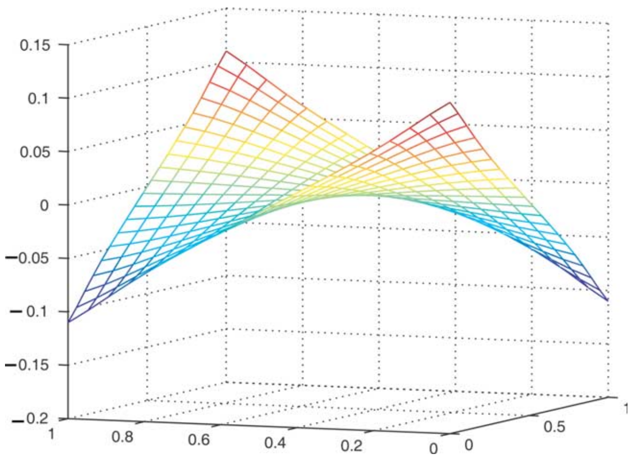

Fig. 11. M_{xy} moment for the plate supported at the corners.

6. Numerical tests—plate bending

A significant set of plate bending tests were already carried out in the course of this line of research. In a previous work [12] several examples were shown that emphasized the generality of the formulation (by applying it to different types of boundary conditions and loads) although by using only regular distribution of collocation points.

In this work, and as a result of the implementation of the previously referred algorithm for the generation of irregular distributions, other types of distributions were tested and the results are quite satisfactory. As a matter of fact, the results show the formulation to be relatively insensitive even to a high degree of irregularity in the distribution of points.

In all the tests of plate bending the following material and normalizing parameters were assumed: $E = 10920.0$, $\nu = 0.3$, $a = 1.0, t = (a/10), \bar{p} = 1.0$ and $\bar{P} = 1.0$, $w^{\text{adim}} = (wD/\bar{p}a^4)$, $M_{ij}^{\text{adim}} = (M_{ij}/\bar{p}a^2)$, $Q_i^{\text{adim}} = (Q_i/\bar{p}a)$.

Consider a clamped square plate subjected to a uniform load of magnitude p as represented in Fig. 4. To assess the effect of randomness in the distribution of points the plate was analysed for three different irregular grids and for a regular grid. In all cases the same number of points was considered, only the distribution varied. The total number of equations in all cases was 128 (Table 1).

The meshes used are represented in Figs. 5–8 where, as mentioned before, parameter k is inversely proportional to the degree of irregularity.

The second test shown here is that of a plate supported at the corners. It is included here to emphasize the applicability of the method even in a relatively difficult situation such as that of a plate supported only at the corners. A total of 93 equations was considered for this example (Figs. 9–12).

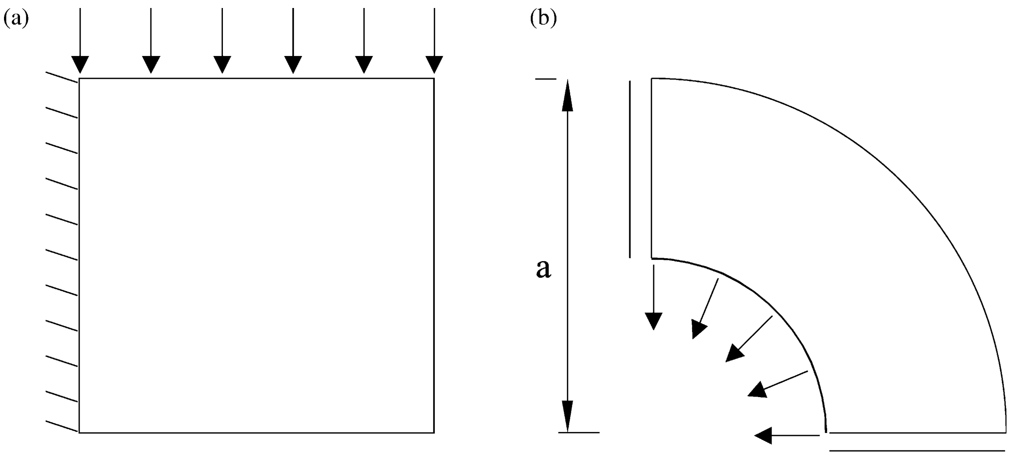

Fig. 13. (a) Square plate. (b) Quarter of thick cylinder

7. Numerical tests—plane states

As this application to plane states is still under development only two examples are shown here. The two examples considered are:

- Square plate clamped on one side subjected to a uniform unit load at the top;
- Quarter of a thick cylinder under unit pressure.

In both cases only regular distribution of collocation points was considered. The same algorithm previously mentioned to create irregular grids will be implemented into this application as well in the near future. A total of 353 equations was considered for the clamped square plate and a total of 153 equations was considered for the quarter of a thick cylinder (Fig. 13).

In Figs. 14 and 15 the stress field components (in the cartesian coordinate system and in the polar coordinate

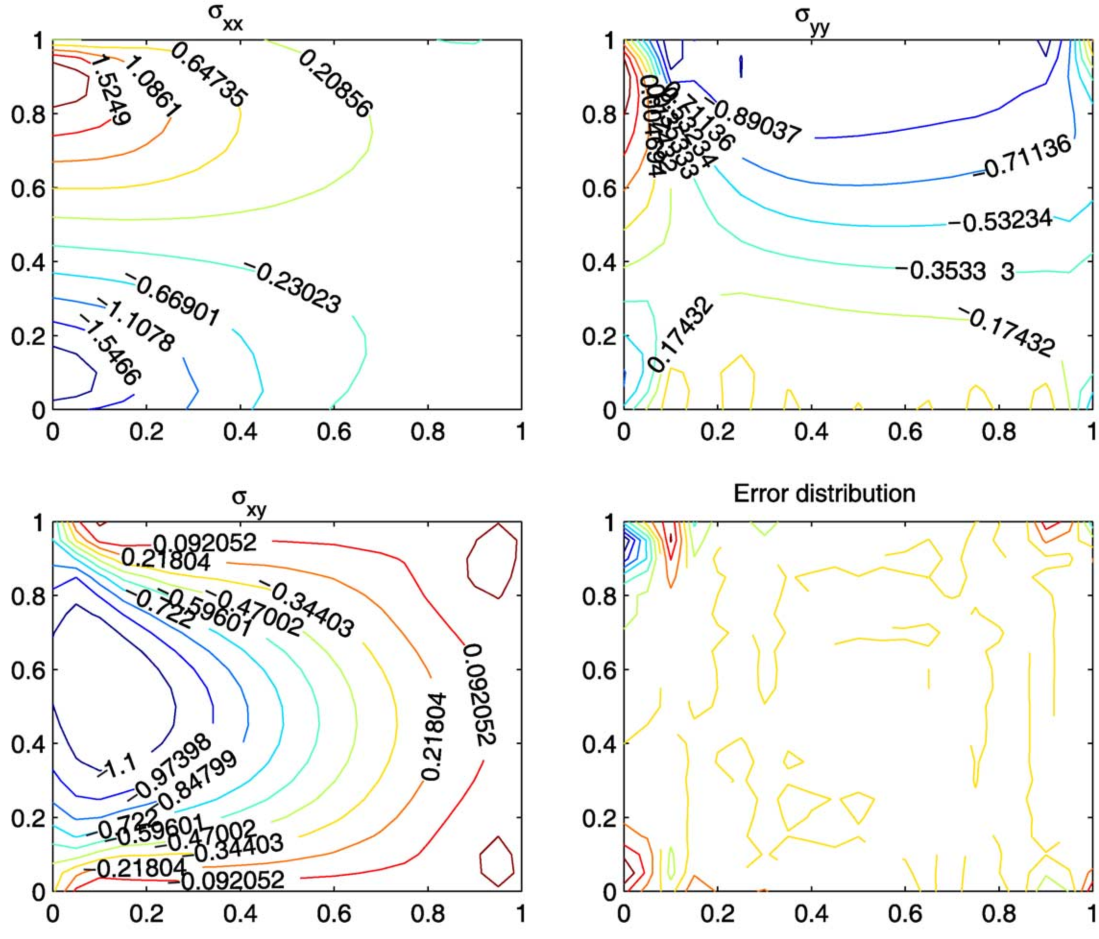

Fig. 14. Stress field components and error distribution for plate clamped on one side and loaded at the top.

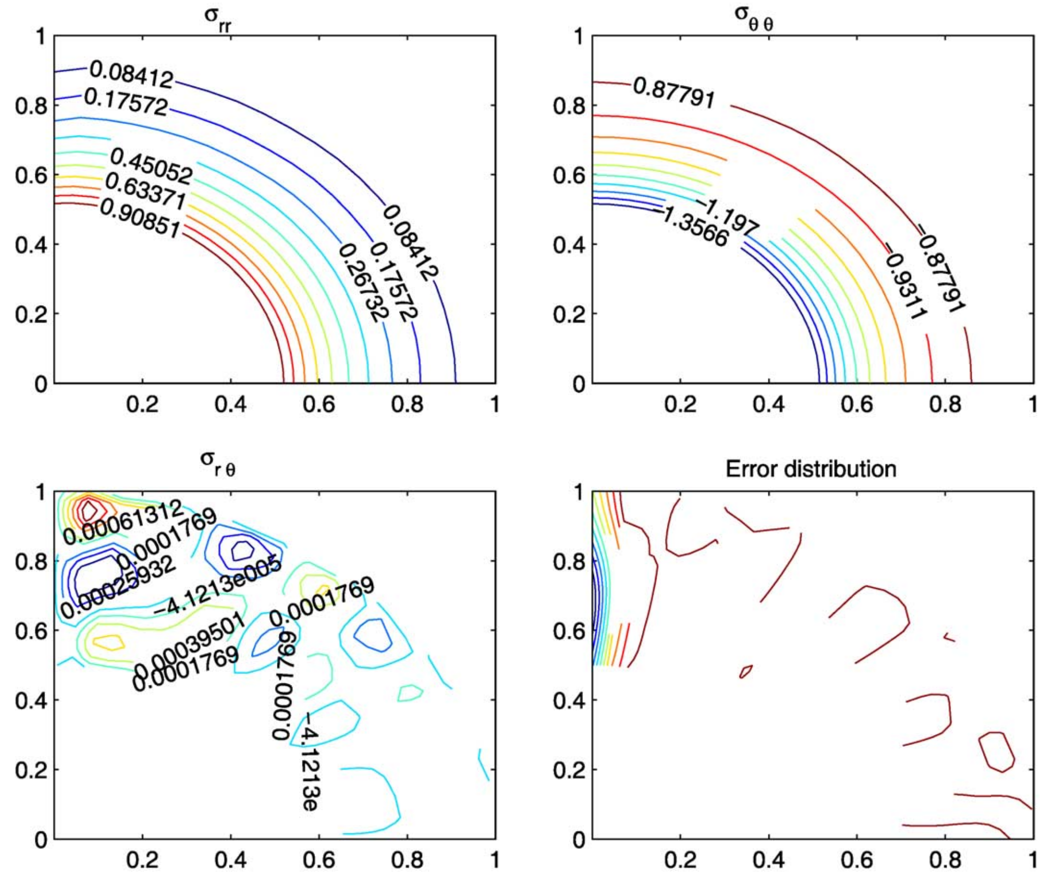

Fig. 15. Stress field components and error distribution for thick cylinder under internal pressure.

system, respectively, for the square plate and the cylinder) are shown together with the distribution of the error, measured in terms of loss of compatibility. These plots are shown here just to illustrate the variations or the shape of the stress fields and this is the reason why a fixed number of isocurves is shown instead of plotting the isocurves at fixed values.

In Fig. 14 notice the higher concentration of error at the top left hand side corner which is not strange since the stresses in that region must accommodate to the conditions imposed by the clamped side and those of the loaded side.

Fig. 15 shows a similar behaviour in the sense that the error is concentrated at the boundary.

8. Discussion and conclusions

This work extends, in two directions, the previous work on the use of RBFs and the Hermite collocation method for Kirchhoff plate bending. The first improvement is that of implementing an algorithm for the generation of irregular distribution of points. This is just a minor improvement but it makes the method a lot more general and promising.

The second improvement is really a whole new application that is still under development. The formulation has yet to be assessed and validated with more examples but the results obtained so far look promising as well.

Generally speaking, the meshless method presented here is based on the use of RBFs to build an approximation of the general solution of the PDEs of a given problem. The approximate solution is obtained by using the Hermite collocation method, that is, by matching, in an appropriate manner, the boundary conditions and the governing equations at selected points.

The main advantages of the proposed technique are its meshless character and the fact that it is conceptually simple. It may also be recognized the computational efficiency of collocation techniques in general and RBF collocation in particular. In fact, other collocation techniques, such as indirect Trefftz collocation techniques [11], are not as efficient as this one especially due to the strong requirements on the approximating functions that have to be used.

A lot of work has still to be made on this subject. Several directions of research are now being exploited by a growing number of researchers in particular on themes such as compactly supported RBFs, variational approaches (Galerkin) using RBFs, multilevel methods and smoothing, domain decomposition amongst others.

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