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Bending of clamped orthotropic rectangular plates: a variational symbolic solution

F.C. Mbakogu[a], M.N. Pavlović[b,*]

[a]Allied Tropical Consultants Ltd, 5 Akinola Cole Crescent, Ikeja, Lagos, Nigeria
[b]Department of Civil Engineering, Imperial College of Science, Technology and Medicine, London SW7 2BU, UK

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Abstract

The Galerkin method is herein applied to the classical bending problem of a uniformly-loaded orthotropic rectangular plate with clamped edges, a problem for which, to our knowledge, no exact analytical solution in its general form exists. The tedious and error-prone computations inherent in such an approach are facilitated through the use of a computer algebra system; and several solutions, based on different approximations for the infinite series representing the assumed deflection function for the plate, are worked out, thereby extending previous work in the literature. The accuracy and convergence of the present formulation are assessed on the basis of solutions corresponding to the special case of material isotropy; and one such existing, manually-derived, result is shown to be incorrect. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Computer algebra (or symbolic manipulation or algebraic computation) systems have become quite popular in engineering analysis [1–11]. These computer programs possess the remarkable capability of manipulating both numbers and symbols; and, as such, they are more versatile than traditional computer languages, like FORTRAN and BASIC, which perform only numerical computations. Without doubt the advent of computer algebra systems has rekindled interest in, and expanded the frontiers of, the classical analytical methods of applied mechanics, which, when practicable, are preferable to the 'discrete' numerical schemes of computation (now commonly associated with the digital computer), for a variety of reasons, including their amenability to parametric studies and the ready insight they afford into the physical aspects of a given problem. The reason for this is that computer-aided algebraic computation can considerably reduce the tedium of analytical calculations while increasing their reliability. An especially useful field of application of computer algebra systems is in finding the solution of the variational problems of mathematical physics, where they enable the possibility of employing higher order approximations in routine fashion [4,5,9,12]. In particular, it has been briefly shown [5] how such an approach could be employed in the treatment, by the well-established variational method

* Corresponding author. Fax: +44-0171-594-5989.
E-mail address: m.pavlovic@ic.ac.uk (M.N. Pavlović).

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Nomenclature

A	surface area of plate
a	plan dimension of plate
$a_{kl}\,(k, l = 0, 1, 2, \ldots)$	numerical coefficients (see Eq. (9))
b	plan dimension of plate
c	b/a
D	flexural rigidity corresponding to an isotropic material
D_1	rigidity defined by Eq. (3)
D_{11}–D_{20}	'product rigidities' (see Eq. (51))
D_x, D_y	flexural rigidities about the y- and x-axes, respectively
D_{xy}	torsional rigidity
H	effective torsional rigidity (see Eq. (2))
k, l	counters in series (9)
M_1–M_{13}	parameters defined by Eqs. (44)–(46)
$M_{fi}\,(i = 1, 2, 3)$	parameters defined by Eqs. (36)– (38)
M_{max}	maximum bending moment per unit length
(M_x, M_y), M_{xy}	bending and twisting moments per unit length, respectively, defined with reference to the (x, y) axes
Q_1–Q_{16}	parameters defined by Eqs. (47)–(50)
$Q_i^*\,(i = 1, 3)$	parameters defined by Eqs. (22) and (34)
Q_{3i}^*	parameter defined by Eq. (56)
Q_{f1}–Q_{f4}	parameters defined by Eqs. (39)–(42)
Q_{max}	maximum transverse shear force per unit length
Q_x, Q_y	transverse shear forces per unit length
q	applied transverse loading
q_0	intensity of uniformly distributed loading
w	transverse displacement function
w_1–w_3	parameters defined by Eq. (43)
w_{f1}	parameter defined by Eq. (35)
w_{kl}	possible deflection function for plate (see Eqs. (9) and (13))
w_{max}	maximum displacement
(x, y)	Cartesian coordinates
ν	Poisson's ratio corresponding to an isotropic material
ν_1, ν_2	reduced Poisson's ratios (see Eq. (3))
$\varphi_{kl}\,(k, l = 0, 1, 2, \ldots)$	displacement functions (see Eq. (9))

of Galerkin, of the bending problem of clamped, isotropic rectangular plates with and without elastic foundations.

It is the purpose of the present article to apply the Galerkin method to the problem of a uniformly-loaded orthotropic rectangular plate with clamped edges, for which, to our knowledge, no exact solution is available. Computations will be carried out by means of Mathematica [13], a versatile computer algebra system capable of performing numerical, symbolic, and graphical computations in a unified manner. Several different approximations for the infinite series representing the deflection function for the orthotropic plate shall be examined in detail, thereby extending previous work reported in the literature, which, so far as we can tell, is limited to the derivation of the deflection function corresponding to a rather crude one-term approximation by means of the (equivalent) Ritz method. The work presented here includes, as a special case, the simpler results corresponding to plates composed of isotropic materials; and one such result reported in Mikhlin's classic treatise [14] will be found incorrect. Moreover, these isotropic-case results will be used as guidelines in assessing the accuracy and convergence of the present formulation.

2. Analytical formulation and results

The governing differential equation (often called Huber's equation [15,16]) for the bending problem of an orthotropic plate for which the principal axes of orthotropy coincide with the x and y directions can be expressed in the following form [15–18]:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{1}$$

in which w and q define the transverse displacement of the plate and the applied transverse loading respectively. The symbols D_x and D_y represent the flexural rigidities about the y- and x-axes, respectively, while the parameter H denotes the effective torsional rigidity [15,16] given by

$$H = D_1 + 2D_{xy} \tag{2}$$

where D_{xy} denotes the torsional rigidity, while the rigidity D_1 is defined in terms of the so-called reduced Poisson's ratios [18] ν_1 andν_2 as

$$D_1 = \nu_2 D_x = \nu_1 D_y \tag{3}$$

Furthermore, the stress couples (M_x, M_y, M_{xy}) and transverse shear forces (Q_x, Q_y) acting on the plate can be calculated from the displacement function w through the following relations [15–18]:

$$M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right) \tag{4}$$

$$M_y = -\left(D_1 \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right) \tag{5}$$

$$M_{xy} = -2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \tag{6}$$

$$Q_x = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 w}{\partial x^2} + H \frac{\partial^2 w}{\partial y^2}\right) \tag{7}$$

$$Q_y = -\frac{\partial}{\partial y}\left(H \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right) \tag{8}$$

Let us assume that the deflection function w can be represented in the form of a series

$$w \approx w_{kl} = \sum_{k=0,\,1,\,2,\ldots}^{\infty} \sum_{l=0,\,1,\,2,\ldots}^{\infty} a_{kl}\, \varphi_{kl} \tag{9}$$

where φ_{kl} denote a complete set of independent, continuous functions suitable for the representation of the deflected surface and the satisfaction of all boundary conditions (but not necessarily of the governing equation (1)); a_{kl} are unknown numerical coefficients. Then the Galerkin (or, more precisely, Bubnov–Galerkin [14]; see also Ref. [19]) formulation of the plate-bending problem for an orthotropic plate of the type under consideration is given in Cartesian coordinates by the following system of equations (the integrals are evaluated over the entire surface area A of the plate, and $\varphi_{kl} = \varphi_{kl}(x, y)$)

$$\int_A \int \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - q\right) \varphi_{00}\, \mathrm{d}x\, \mathrm{d}y = 0$$

$$\int_A \int \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - q\right) \varphi_{10}\, \mathrm{d}x\, \mathrm{d}y = 0$$

$$\vdots$$

$$\int_A \int \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - q\right) \varphi_{kl}\, d\mathrm{x}\, \mathrm{d}y = 0 \tag{10}$$

Thus, the problem can essentially be solved by substituting expression (9) into Eq. (10) and solving the resulting linear equation system for the unknown coefficients a_{kl}; once the latter have been calculated, the (approximate) response of the plate can be determined explicitly through Eq. (9).

In the special case of an isotropic medium, the following relations hold true [17,18]

$$\nu_1 = \nu_2 = \nu; \quad D_x = D_y = H = D; \quad D_1 = \nu D; \quad D_{xy} = D\frac{1-\nu}{2} \tag{11}$$

where the symbols D and ν represent the flexural rigidity and Poisson's ratio of the material of the plate. Clearly, for such a plate, the foregoing formulation is

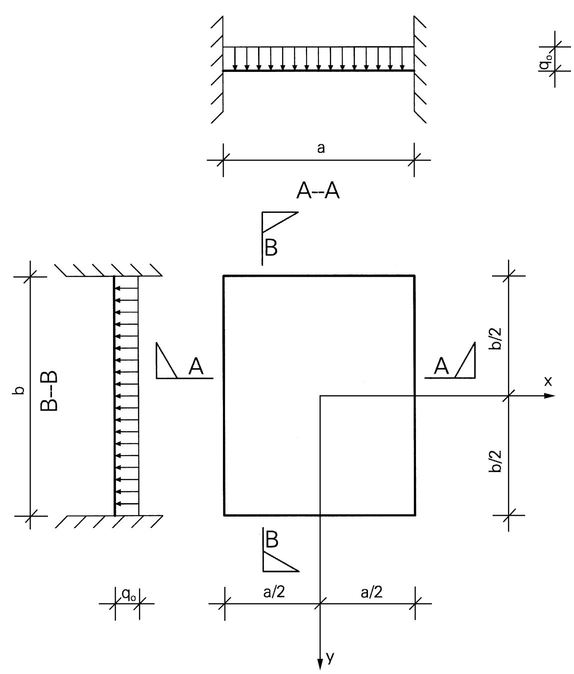

Fig. 1. Uniformly-loaded, clamped rectangular plate.

considerably simplified. In particular, Eq. (1) reduces to the well-known governing equation for isotropic plates [15–17,20] while the Galerkin equation system (10) corresponds to the simpler system of equations presented in Ref. [17].

Now consider the specific case of a uniformly-loaded rectangular plate with constant thickness and all four edges clamped (as shown in Fig. 1, the loading intensity is q_0); for such a plate the boundary conditions are

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at}\, x = \pm\frac{a}{2}$$

$$w = \frac{\partial w}{\partial y} = 0 \quad \text{at}\, y = \pm\frac{b}{2} \tag{12}$$

Clearly, the following infinite series (cf. Refs. [14,17,18])

$$w \approx w_{kl} = \left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2 \sum_{k=0,\,1,\,2,\ldots}^{\infty} \sum_{l=0,\,1,\,2,\ldots}^{\infty} a_{kl}\, x^k\, y^l \tag{13}$$

is suitable for the present analysis, since it comprises a set of coordinate functions which individually satisfy the boundary conditions; for practical purposes, however, it is sensible to omit odd powers of x and y in the above series (13), as demanded by the inherent symmetry of the deflection function w about the coordinate axes.

In what follows we shall consider three different approximations for the series (13), thus extending previous work reported in the literature [16,18], which, to our knowledge, is limited to the derivation of the deflection function corresponding to a one-term approximation by means of the (equivalent) Ritz method. Included in the present work are corresponding expressions for stress couples and transverse shear forces, defined by Eqs. (4)–(8). Also, the simplified versions of the above results corresponding to an isotropic plate are given, thereby generalizing earlier solutions [14,17] for such plates: namely, for this special case of material isotropy, Timoshenko and Woinowsky-Krieger [17] presented solutions for the deflection function of a square plate corresponding to both a one-term and a four-term (strictly, three-term, because of diagonal symmetry) approximation, based on the Galerkin method (including numerical values of bending moments (M_x, M_y) at certain locations, obtained by means of the above deflection functions); furthermore, Mikhlin [14] used the Ritz method to derive the corresponding one-term solution for a rectangular plate and the three-term (strictly two-term) solution for a square plate but without calculating the associated stress couples and transverse shear forces. The isotropic-case solutions corresponding to the present formulation are, thanks to the use of symbolic computation, not only precise (with fractions rather than decimals), but also tidier than the previous solutions; they are also more general than existing data, for they apply to rectangular plates (having arbitrary aspect ratios ($c = b/a;\, c \geq 1$)) and also encompass all the structural-response parameters for a given problem. The presently obtained isotropic-case results are used to assess the accuracy/convergence of the current formulation; the benchmark analytical solutions for isotropy which form the basis for this assessment are the approximate series solutions worked out by Wojtaszak [21], Evans [22] and Young [23], through the application of the so-called *superposition methods* proposed by Hencky [24] and Timoshenko [25] (see also Refs. [17,19]).

2.1. First approximation

For this approximation the deflection function (13) is assumed to take the following form

$$w = a_{00}\varphi_{00} \tag{14}$$

where

$$\varphi_{00} = \left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2 \tag{15}$$

Clearly, the problem is basically solved by substituting the above expression (14) into the first of Eq. (10), with the integrals evaluated over the range $x = \pm a/2,\, y = \pm b/2$. Once the coefficient a_{00} is obtained from the resulting equation, the (approximate) plate response can be worked out (using Eq. (14)). The Mathematica procedure adopted for this computation, which includes expressions for the determination of the associated moment and shear-force fields, is given in Appendix A; and the results are

$$w = \frac{49q_0}{2048a^4Q_1^*}(4x^2 - a^2)^2(4y^2 - a^2c^2)^2 \tag{16}$$

$$M_x = \frac{49q_0}{128a^4Q_1^*}\Big[D_1(4x^2 - a^2)^2(a^2c^2 - 12y^2) + D_x(a^2 - 12x^2)(4y^2 - a^2c^2)^2\Big] \tag{17}$$

$$M_y = \frac{49q_0}{128a^4Q_1^*}\Big[D_y(4x^2 - a^2)^2(a^2c^2 - 12y^2) + D_1(a^2 - 12x^2)(4y^2 - a^2c^2)^2\Big] \tag{18}$$

$$M_{xy} = -\frac{49q_0}{4a^4Q_1^*}D_{xy}xy(4x^2 - a^2)(4y^2 - a^2c^2) \quad (19)$$

$$Q_x = \frac{49q_0}{16a^4Q_1^*}x\big[3D_x(8a^2c^2y^2 - 16y^4 - a^4c^4) + 2H(4a^2c^2x^2 - 48x^2y^2 + 12a^2y^2 - a^4c^2)\big] \quad (20)$$

$$Q_y = \frac{49q_0}{16a^4Q_1^*}y\Big[3D_y(8a^2x^2 - 16x^4 - a^4) + 2H(12a^2c^2x^2 - 48x^2y^2 + 4a^2y^2 - a^4c^2)\Big] \quad (21)$$

where

$$Q_1^* = 7c^4D_x + 4c^2H + 7D_y \quad (22)$$

It can easily be shown that the maximum deflection $w_{\max}$ (at the centre $(x = 0, y = 0)$), bending moment $M_{\max}$ (corresponding to M_x at the edge point $x = a/2, y = 0$) and shear force $Q_{\max}$ (corresponding to Q_x at $x = a/2, y = 0$) are given by the following expressions:

$$w_{\max} = \frac{49q_0a^4c^4}{2048Q_1^*} \quad (23)$$

$$M_{\max} = -\frac{49q_0a^2c^4D_x}{64Q_1^*} \quad (24)$$

$$Q_{\max} = -\frac{147q_0ac^4D_x}{32Q_1^*} \quad (25)$$

Furthermore, one can readily obtain the corresponding formulae for an isotropic plate from the foregoing Eqs. (16)–(25) by setting $D_x = D_y = H = D$; $D_1 = \nu D$; $D_{xy} = D(1 - \nu)/2$ (see Eq. (11)).

As noted earlier, the above formulation for a rectangular orthotropic plate was previously considered in Refs. [16,18], where Eq. (16) for the displacement function and its isotropic counterpart are presented; the latter result alone can also be found in Refs. [14,26] (the above works are all based on the equivalent Ritz method of solution). On the other hand, Timoshenko and Woinowsky-Krieger [17], using the Galerkin technique adopted here, treated the special case of a square isotropic plate, giving numerical values of the displacement function and bending moments at two specific locations on the plate (viz, w and $M_x = M_y$ at $x = 0, y = 0$; and M_x at $x = a/2, y = 0$). In the present work, by contrast, the displacement function together with the stress couples and transverse shear forces have been presented in general form, and this enables a thorough assessment of the accuracy of the solution to be easily achieved. This has indeed been carried out, as indicated in Tables 1–3, by reference to the isotropic-case problem and on the basis of the analytical series solutions mentioned earlier.

It can be seen that the order of accuracy obtainable from this first approximation is rather poor, although the response pattern is correctly predicted. For the (maximum) deflection values displayed in Table 1, the greatest error (roughly, 31.5%) in the present solution occurs in the limiting case of infinite aspect ratio ($c = \infty$); however, the error is considerably less for plates having aspect ratios $1.0 \le c \le 2.0$, ranging from 5.6 (for $c = 1.0$) to 11.8% (for $c = 2.0$) approximately. (It may be useful to note that values of displacement, stress couples, and shear forces corresponding to the case of infinite aspect ratio can readily be obtained by computing their limiting values as $c \to \infty$, using the Mathematica built-in function **Limit**.) As would be expected, the displacement field is predicted to a higher degree of accuracy than the moment field, which, in turn, is predicted to a greater level of accuracy than the shear-force field. This is, of course, due to the fact that the stress couples and transverse shear forces are proportional to the second and third derivatives of the displacement function, respectively. Thus, for plates having $1.0 \le c \le 2.0$, the errors in the (maximum) bending moment M_x at $x = a/2, y = 0$ (Table 2) range from 0.4 (for $c = 1.5$) to 17.2% (for $c = 1.0$) approximately; the corresponding errors for the (maximum) shear force Q_x at the same location (based on the smaller range of results quoted in Table 3) are 1.9 (for $c = 1.75$) to 40.9% (for $c = 1.0$). (Note that the errors in the calculated values of M_x and Q_x (at $x = a/2, y = 0$) corresponding to the limiting case $c = \infty$ are roughly the same as their counterpart (31.5%) for the displacement at the center, viz: 31.3% for M_x; 32.0% for Q_x.)

2.2. Second approximation

Here we assume a three-term approximation for the deflection (13),as follows

$$w = a_{00}\varphi_{00} + a_{20}\varphi_{20} + a_{02}\varphi_{02} \quad (26)$$

in which φ_{00} is defined by Eq. (15), while

$$\varphi_{20} = \left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2x^2$$

$$\varphi_{02} = \left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2y^2 \quad (27)$$

By combining the above expression (26) with Eq. (10) and solving the ensuing 3 × 3 equation system, the coefficients a_{00}, a_{20}, and a_{02} can be worked out; and

Table 1
Variation of non-dimensional displacement $w(q_0a^4/D)$ at the centre ($x = 0$, $y = 0$) with aspect ratio $c(= b/a)$, for an isotropic plate, and comparison with the benchmark series solution [22] (see also Refs. [17,21,23])

c	Present solution			Series solution [22]
	First approximation	Second approximation	Third approximation	
1.0	0.00133	0.00126	0.00127	0.00126
1.1	0.00159	0.00150	0.00151	0.00150
1.2	0.00182	0.00172	0.00173	0.00172
1.3	0.00202	0.00191	0.00191	0.00191
1.4	0.00220	0.00206	0.00207	0.00207
1.5	0.00235	0.00219	0.00220	0.00220
1.6	0.00248	0.00228	0.00230	0.00230
1.7	0.00259	0.00236	0.00238	0.00238
1.8	0.00269	0.00242	0.00245	0.00245
1.9	0.00277	0.00246	0.00250	0.00249
2.0	0.00284	0.00249	0.00254	0.00254
∞	0.00342	0.00215	0.00291	0.00260

once this is done all the response parameters readily follow. A suitable Mathematica procedure for this exercise is displayed in Appendix B. The corresponding results are

$$w = \frac{77q_0}{2048a^6Q_3^*}(4x^2 - a^2)^2\left(4y^2 - a^2c^2\right)^2 w_{f1} \tag{28}$$

$$M_x = \frac{77q_0}{256a^6Q_3^*}\left[\left(4y^2 - a^2c^2\right)^2 D_x M_{f1} - (4x^2 - a^2)^2 D_1 M_{f2}\right] \tag{29}$$

$$M_y = \frac{77q_0}{256a^6Q_3^*}\left[\left(4y^2 - a^2c^2\right)^2 D_1 M_{f1} - (4x^2 - a^2)^2 D_y M_{f2}\right] \tag{30}$$

$$M_{xy} = \frac{77q_0}{8a^6Q_3^*}D_{xy}xy(4x^2 - a^2)\left(4y^2 - a^2c^2\right)M_{f3} \tag{31}$$

$$Q_x = -\frac{77q_0}{16a^6Q_3^*}x\left[3D_x\left(4y^2 - a^2c^2\right)^2 Q_{f1} - H(a^2 - 4x^2)Q_{f2}\right] \tag{32}$$

Table 2
Variation of non-dimensional bending moment$M_x(q_0a^2)$ at the edge point $x = a/2$, $y = 0$ with aspect ratio $c(= b/a)$, for an isotropic plate, and comparison with the benchmark series solution [22] (see also Refs. [17,21,23])

c	Present solution			Series solution [22]
	First approximation	Second approximation	Third approximation	
1.0	−0.0425	−0.0521	−0.0515	−0.0513
1.1	−0.0507	−0.0592	−0.0582	−0.0581
1.2	−0.0582	−0.0653	−0.0640	−0.0639
1.3	−0.0648	−0.0704	−0.0688	−0.0687
1.4	−0.0705	−0.0744	−0.0726	−0.0726
1.5	−0.0754	−0.0776	−0.0756	−0.0757
1.6	−0.0795	−0.0801	−0.0779	−0.0780
1.7	−0.0830	−0.0819	−0.0796	−0.0799
1.8	−0.0860	−0.0831	−0.0809	−0.0812
1.9	−0.0886	−0.0840	−0.0817	−0.0822
2.0	−0.0907	−0.0844	−0.0823	−0.0829
∞	−0.1094	−0.0688	−0.0931	−0.0833

$$Q_y = \frac{77q_0}{16a^6 Q_3^*} y\Big[3D_y(4x^2 - a^2)^2 Q_{f3} + H\big(a^2c^2 - 4y^2\big)Q_{f4}\Big] \quad (33)$$

in which

$$Q_3^* = 25{,}025\Big(c^{12}D_x^3 + D_y^3\Big) + 834{,}635c^4\big(c^4D_{14} + D_{15}\big) + 129{,}740c^2\big(c^8D_{16} + D_{17}\big) + 718{,}608c^6D_{18} + 77{,}363c^4\big(c^4D_{19} + D_{20}\big) + 7436c^6H^3 \quad (34)$$

$$w_{f1} = a^2w_1 + x^2w_2 + y^2w_3 \quad (35)$$

$$M_{f1} = a^4M_1 + a^2x^2M_2 + x^4M_3 + a^2y^2M_4 + x^2y^2M_5 \quad (36)$$

$$M_{f2} = a^4M_6 + a^2x^2M_7 + a^2y^2M_8 + x^2y^2M_9 + y^4M_{10} \quad (37)$$

$$M_{f3} = a^2M_{11} + x^2M_{12} + y^2M_{13} \quad (38)$$

$$Q_{f1} = a^2Q_1 + x^2Q_2 + y^2Q_3 \quad (39)$$

$$Q_{f2} = a^4Q_4 + a^2x^2Q_5 + a^2y^2Q_6 + x^2y^2Q_7 + y^4Q_8 \quad (40)$$

$$Q_{f3} = a^2Q_9 + x^2Q_{10} + y^2Q_{11} \quad (41)$$

$$Q_{f4} = a^4Q_{12} + a^2x^2Q_{13} + x^4Q_{14} + a^2y^2Q_{15} + x^2y^2Q_{16} \quad (42)$$

where

$$w_1 = 1430\Big(c^8D_x^2 + D_y^2\Big) + 73{,}036c^4D_{11} + 9477c^2\big(c^4D_{12} + D_{13}\big) + 1183c^4H^2 \quad (43a)$$

$$w_2 = 1404c^4D_{11} + 37{,}180D_y^2 + 1144c^6D_{12} + 64{,}220c^2D_{13} + 7436c^4H^2 \quad (43b)$$

$$w_3 = 37{,}180c^6D_x^2 + 1404c^2D_{11} + 64{,}220c^4D_{12} + 1144D_{13} + 7436c^2H^2 \quad (43c)$$

$$M_1 = 2860c^8D_x^2 + 145{,}721c^4D_{11} - 6435D_y^2 + 18{,}668c^6D_{12} + 2899c^2D_{13} + 507c^4H^2 \quad (44a)$$

$$M_2 = -34{,}320c^8D_x^2 - 1{,}736{,}016c^4D_{11} + 411{,}840D_y^2 - 213{,}720c^6D_{12} + 543{,}192c^2D_{13} + 60{,}840c^4H^2 \quad (44b)$$

$$M_3 = -\Big(84{,}240c^4D_{11} + 2{,}230{,}800D_y^2 + 68{,}640c^6D_{12} + 3{,}853{,}200c^2D_{13} + 446{,}160c^4H^2\Big) \quad (44c)$$

Table 3
Variation of non-dimensional shear force $Q_x(q_0a)$ at the edge point $x = a/2, y = 0$ with aspect ratio $c(= b/a)$, for an isotropic plate, and comparison with the benchmark series solution [21] (see also Refs. [19,23])

c	Present solution			Series solution [21]
	First approximation	Second approximation	Third approximation	
1.00	−0.26	−0.45	−0.45	−0.44
1.25	−0.37	−0.53	−0.50	−0.49
1.50	−0.45	−0.56	−0.52	−0.52
1.75	−0.51	−0.57	−0.52	−0.52
2.00	−0.54	−0.56	−0.51	−0.52
∞	−0.66	−0.41	−0.56	−0.50

$$M_4 = 74{,}360c^6D_x^2 + 2808c^2D_{11} + 128{,}440c^4D_{12} + 2288D_{13} + 14{,}872c^2H^2 \tag{44d}$$

$$M_5 = -\left(892{,}320c^6D_x^2 + 33{,}696c^2D_{11} + 1{,}541{,}280c^4D_{12} + 27{,}456D_{13} + 178{,}464c^2H^2\right) \tag{44e}$$

$$M_6 = 6435c^{10}D_x^2 - 145{,}721c^6D_{11} - 2860c^2D_y^2 - 2899c^8D_{12} - 18{,}668c^4D_{13} - 507c^6H^2 \tag{45a}$$

$$M_7 = -\left(2808c^6D_{11} + 74{,}360c^2D_y^2 + 2288c^8D_{12} + 128{,}440c^4D_{13} + 14{,}872c^6H^2\right) \tag{45b}$$

$$M_8 = -411{,}840c^8D_x^2 + 1{,}736{,}016c^4D_{11} + 34{,}320D_y^2 - 543{,}192c^6D_{12} + 213{,}720c^2D_{13} - 60{,}840c^4H^2 \tag{45c}$$

$$M_9 = 33{,}696c^4D_{11} + 892{,}320D_y^2 + 27{,}456c^6D_{12} + 1{,}541{,}280c^2D_{13} + 178{,}464c^4H^2 \tag{45d}$$

$$M_{10} = 2{,}230{,}800c^6D_x^2 + 84{,}240c^2D_{11} + 3{,}853{,}200c^4D_{12} + 68{,}640D_{13} + 446{,}160c^2H^2 \tag{45e}$$

$$M_{11} = 6435\left(c^8D_x^2 + D_y^2\right) - 145{,}370c^4D_{11} - 2613c^2\left(c^4D_{12} + D_{13}\right) + 1352c^4H^2 \tag{46a}$$

$$M_{12} = -\left(4212c^4D_{11} + 111{,}540D_y^2 + 3432c^6D_{12} + 192{,}660c^2D_{13} + 22{,}308c^4H^2\right) \tag{46b}$$

$$M_{13} = -\left(111{,}540c^6D_x^2 + 4212c^2D_{11} + 192{,}660c^4D_{12} + 3432D_{13} + 22{,}308c^2H^2\right) \tag{46c}$$

$$Q_1 = 1430c^8D_x^2 + 72{,}334c^4D_{11} - 17{,}160D_y^2 + 8905c^6D_{12} - 22{,}633c^2D_{13} - 2535c^4H^2 \tag{47a}$$

$$Q_2 = 7020c^4D_{11} + 185{,}900D_y^2 + 5720c^6D_{12} + 321{,}100c^2D_{13} + 37{,}180c^4H^2 \tag{47b}$$

$$Q_3 = 37{,}180c^6D_x^2 + 1404c^2D_{11} + 64{,}220c^4D_{12} + 1144D_{13} + 7436c^2H^2 \tag{47c}$$

$$Q_4 = 6435c^2\left(c^8D_x^2 + D_y^2\right) - 145{,}370c^6D_{11} - 2613c^4\left(c^4D_{12} + D_{13}\right) + 1352c^6H^2 \tag{48a}$$

$$Q_5 = -\left(4212c^6D_{11} + 111{,}540c^2D_y^2 + 3432c^8D_{12} + 192{,}660c^4D_{13} + 22{,}308c^6H^2\right) \tag{48b}$$

$$Q_6 = -411{,}840c^8D_x^2 + 1{,}731{,}804c^4D_{11} - 77{,}220D_y^2 - 546{,}624c^6D_{12} + 21{,}060c^2D_{13} - 83{,}148c^4H^2 \tag{48c}$$

$$Q_7 = 50{,}544c^4D_{11} + 1{,}338{,}480D_y^2 + 41{,}184c^6D_{12} + 2{,}311{,}920c^2D_{13} + 267{,}696c^4H^2 \tag{48d}$$

$$Q_8 = 2{,}230{,}800c^6D_x^2 + 84{,}240c^2D_{11} + 3{,}853{,}200c^4D_{12} + 68{,}640D_{13} + 446{,}160c^2H^2 \tag{48e}$$

$$Q_9 = 17{,}160c^8D_x^2 - 72{,}334c^4D_{11} - 1430D_y^2 + 22{,}633c^6D_{12} - 8905c^2D_{13} + 2535c^4H^2 \tag{49a}$$

$$Q_{10} = -\left(1404c^4D_{11} + 37{,}180D_y^2 + 1144c^6D_{12} + 64{,}220c^2D_{13} + 7436c^4H^2\right) \tag{49b}$$

$$Q_{11} = -\left(185{,}900c^6D_x^2 + 7020c^2D_{11} + 321{,}100c^4D_{12} + 5720D_{13} + 37{,}180c^2H^2\right) \quad (49c)$$

$$Q_{12} = 6435\left(c^8D_x^2 + D_y^2\right) - 145{,}370c^4D_{11} - 2613c^2\left(c^4D_{12} + D_{13}\right) + 1352c^4H^2 \quad (50a)$$

$$Q_{13} = -77{,}220c^8D_x^2 + 1{,}731{,}804c^4D_{11} - 411{,}840D_y^2 + 21{,}060c^6D_{12} - 546{,}624c^2D_{13} - 83{,}148c^4H^2 \quad (50b)$$

$$Q_{14} = 84{,}240c^4D_{11} + 2{,}230{,}800D_y^2 + 68{,}640c^6D_{12} + 3{,}853{,}200c^2D_{13} + 446{,}160c^4H^2 \quad (50c)$$

$$Q_{15} = -\left(111{,}540c^6D_x^2 + 4212c^2D_{11} + 192{,}660c^4D_{12} + 3432D_{13} + 22{,}308c^2H^2\right) \quad (50d)$$

$$Q_{16} = 1{,}338{,}480c^6D_x^2 + 50{,}544c^2D_{11} + 2{,}311{,}920c^4D_{12} + 41{,}184D_{13} + 267{,}696c^2H^2 \quad (50e)$$

and the following relations define 'product rigidities' in the above Eqs. (34) and (43)–(50):

$$D_{11} = D_xD_y; \quad D_{12} = D_xH; \quad D_{13} = D_yH; \quad D_{14} = D_x^2D_y; \quad D_{15} = D_xD_y^2;$$

$$D_{16} = D_x^2H; \quad D_{17} = D_y^2H; \quad D_{18} = D_xD_yH; \quad D_{19} = D_xH^2; \quad D_{20} = D_yH^2 \quad (51)$$

The maximum deflection, bending moment, and shear force corresponding to the present approximation can readily be shown to be given by the following expressions:

$$w_{\max} = \frac{77q_0a^4c^4}{2048Q_3^*}\left[1430\left(c^8D_x^2 + D_y^2\right) + 73{,}036c^4D_{11} + 9477c^2\left(c^4D_{12} + D_{13}\right) + 1183c^4H^2\right] \quad (52)$$

$$M_{\max} = -\frac{77q_0a^2c^4}{64Q_3^*}\left(1430c^8D_x^3 + 73{,}387c^4D_{14} + 10{,}725D_{15} + 9763c^6D_{16} + 25{,}532c^2D_{18} + 3042c^4D_{19}\right) \quad (53)$$

$$Q_{\max} = -\frac{231q_0ac^4}{32Q_3^*}\left(1430c^8D_x^3 + 74{,}089c^4D_{14} + 29{,}315D_{15} + 10{,}335c^6D_{16} + 57{,}642c^2D_{18} + 6760c^4D_{19}\right) \quad (54)$$

The special case of an isotropic plate is readily attained by applying the relevant relations (11) to the foregoing equations. In particular, the deflection function for such a plate can easily be shown to be given by the following expression

$$w = \frac{77q_0}{2048a^6DQ_{3i}^*}(4x^2 - a^2)^2\left(4y^2 - a^2c^2\right)^2 \times \Big\{a^2\left[1430\left(1 + c^8\right) + 9477c^2\left(1 + c^4\right) + 74{,}219c^4\right] + x^2\left(37{,}180 + 64{,}220c^2 + 8840c^4 + 1144c^6\right) + y^2\left(1144 + 8840c^2 + 64{,}220c^4 + 37{,}180c^6\right)\Big\} \quad (55)$$

in which

$$Q_{3i}^* = 25{,}025\left(1 + c^{12}\right) + 129{,}740c^2\left(1 + c^8\right) + 911{,}998c^4\left(1 + c^4\right) + 726{,}044c^6 \quad (56)$$

Furthermore, it is a trivial exercise to show that, for a square (isotropic) plate ($c = 1$), the deflection function (55) reduces to the following form

$$w = \frac{77q_0}{16{,}404{,}480a^6D}(4x^2 - a^2)^2\left(4y^2 - a^2\right)^2\left[269a^2 + 312\left(x^2 + y^2\right)\right] \quad (57)$$

It is important to note that an *incorrect* expression corresponding to the above result (57) has been presented in Mikhlin's classic treatise [14]; the error in the above work must have been introduced in the course of solving the 3×3 Ritz equation system corresponding to the square plate considered, for we found the above equation system (which can be reduced to 2×2 form as two of the equations are the same on account of diagonal symmetry) to be, in fact, correct (and, of course, equivalent to its Galerkin counterpart implicit in the present formulation). In this connection, one may also note that an expression for the (maximum) deflection at the centre of a square plate corresponding to the incorrect solution presented by Mikhlin [14] has been quoted by Dym and Shames [26]; this *incorrect* Ritz-formulation-based expression — the source of, or calculation for, which was not indicated by the above writers — may have been taken from Ref. [14]. Clearly, this illustrates the usefulness of symbolic-manipulation systems in reliably performing tedious calculations of the kind encountered in the present solution process, as well as in checking existing, manually-derived, results in the literature, as demonstrated elsewhere [27].

As would be expected, the present approximation yields more accurate results than the preceding, one-term, approximation. We find from Table 1, for example, that for plates having aspect ratios $1.0 \leq c \leq 2.0$, the errors in the calculated displacements range (roughly) from 0.5 to 2.0%, and that, for a number of the aspect ratios considered herein, these displacement results coincide with their benchmark counterparts (at least for the number of decimal places to which both sets of results have been evaluated). Evidently, the calculated bending moments and transverse shear forces corresponding to $1.0 \leq c \leq 2.0$ (Tables 2 and 3) are not as accurate as their displacement counterparts, although they represent a marked improvement over the corresponding actions for the one-term approximation. Thus, for the results quoted in Tables 2 and 3 for plates having aspect ratios $1.0 \leq c \leq 2.0$, the maximum errors in the bending moment and shear force are now reduced to 2.7% (for $c = 1.6$) and 9.6% (for $c = 1.75$) (note that their counterparts for the first approximation are 17.2% (for M_x) and 40.9% (for Q_x), both values corresponding to the square plate).As indicated in Tables 1–3, the maximum error for each of the three calculated response parameters corresponds to the limiting case $c = \infty$ (c.f. preceding approximation), the errors varying from 17.3 to 18.0%. Taken together, the above results indicate that the present approximation can be applied with reasonable accuracy to practical plates, at least for purposes of preliminary design.

2.3. Third approximation

For this approximation the series (13) is assumed to be comprised of six terms, as follows

$$w = a_{00}\varphi_{00} + a_{20}\varphi_{20} + a_{02}\varphi_{02} + a_{40}\varphi_{40} + a_{04}\varphi_{04} + a_{22}\varphi_{22} \tag{58}$$

where φ_{00}, φ_{20}, and φ_{02} are defined by Eqs. (15) and (27), while

$$\varphi_{40} = \left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2 x^4$$

$$\varphi_{04} = \left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2 y^4$$

$$\varphi_{22} = \left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2 x^2y^2 \tag{59}$$

The unknown coefficients a_{kl} in Eq. (58) can, of course, be determined by solving the 6×6 algebraic equation system obtained by combining the above expression (58) with Eq. (10); the ensuing results — which can readily follow from a Mathematica procedure similar to that shown in Appendix B for the preceding (i.e. three-term) approximation — are quite voluminous, and, for reasons of brevity, have not been presented here.

Highly accurate results can be obtained by means of the present approximation, with the maximum errors in the calculated values of the response parameters (including shear forces) displayed in Tables 1–3 being of the order of 2% for plates having aspect ratios $1.0 \leq c \leq 2.0$ (note, in particular, that the displacement field is practically indistinguishable from its counterpart predicted by the benchmark solution [22], with the two sets of results coinciding – for the number of decimal places quoted – for the majority of the aspect ratios presently considered). Once more, the greatest error for each of the parameters in Tables 1–3 corresponds to the limiting case of infinite aspect ratio ($c = \infty$), the errors being, roughly, 12%. Obviously, higher order approximations would be required to reduce the errors corresponding to this limiting case to within values acceptable for engineering purposes; but, as indicated by the foregoing results, such higher order approximations would clearly be unnecessary for the treatment of practical plate problems since, as is well known, beyond aspect ratios of $c \approx 2$, one-way beam bending (along the shorter side) predominates.

3. Conclusions

The well-known Galerkin method has been applied in the solution of the classical bending problem of a uniformly-loaded clamped rectangular plate endowed with orthotropic material properties. The tedious, time-consuming and error-prone computations inherent in such an approach have been facilitated through the use of Mathematica, a versatile computer algebra system capable of performing numerical, symbolic and graphical calculations in a unified manner. It is worth noting, however, that the present investigation could, alternatively, be carried out — with similar results — by means of any one of a number of other available computer algebra systems (such as, for example, Derive, Macsyma, Maple and Reduce [4,12,28]), which are also capable of performing the rather basic algebraic computations involved in the formulation.

Three different approximations (having one, three and six terms, respectively) for the infinite series representing the assumed deflection function for a plate of the type considered are treated, thereby extending previous published work, which, to our knowledge, is limited to the derivation of the deflection function corresponding to a one-term approximation by means of the (equivalent) Ritz method. Evidently, the present work includes, as a special case, the simpler results pertaining to plates composed of isotropic materials, some of which have — thanks to the current use of algebraic computation — now been generalized. Moreover, these isotropic-case results have been used to advantage in assessing the accuracy and convergence of the present formulation.

It is worth noting that the closed-form variational approach given here can readily be applied to rectangular plates, having arbitrary aspect ratios and material properties, in routine fashion. As would be expected, the requisite calculations and the ensuing output increase as the number of terms included in the series for the deflection function increases (albeit with concomitant improvement in the accuracy of the predicted response). Clearly, it should, in theory, be easy to implement higher order approximations for the deflection function, leading to further improvement in accuracy. In practice, however, it is found that the "exponential" growth of symbolic computation as the number of independent parameters is augmented [3,7,9,12] effectively limits the size of the problem that can be solved. Indeed, we found that, for problems of the type presently considered, deflection functions having more than six terms could not be implemented, owing to insufficient computer memory capacity. Nevertheless, this shortcoming of the present approach is tempered by the fact that such higher-order approximations have been shown to be, as a rule, accurate enough for the treatment of practical problems. Furthermore, it is worthy of note that the solutions given here are ideally suited to parametric studies: thus, for example, it is obvious that the specialized versions of the present results corresponding to isotropic plates are more amenable to such studies than the previous analytical series solutions [21–23] used presently for benchmarking purposes, which have been formally solved only for specific values of plate geometries c and a single value of Poisson's ratio (namely, $\nu = 0.3$).

Extensions and generalizations of the present approach to the calculation of plates are easily envisaged. For instance, the present solution scheme can readily be extended so as to encompass plates resting on elastic foundations (as noted earlier, the treatment of such problems, in the context of material isotropy, has already been briefly discussed in Ref. [5]): essentially, the scheme of calculation can proceed as in the case of ordinary plates (i.e. those without elastic foundations) once the Galerkin Eqs. (10) have been suitably modified in order to include the reaction of the foundation under the externally applied loading. Also, generalizations of the present approach to include problems involving plates having other than rectangular geometries (e.g. elliptical, circular, and triangular shapes) and clamped boundaries, as well as those pertaining to non-uniform loadings, pose no conceptual difficulties; and, indeed, solutions to some such problems, based on the present technique and other variational methods, can be found in the literature [5,16–18]. Clearly, the application of computer algebra not only facilitates the solution of these and more difficult problems (such as those involving elastic foundations), but also leads to the obtention of more accurate results, by enabling the implementation of higher order approximations in a straightforward manner.

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Appendix A. Mathematica procedure for the first approximation

```
(* Specify the assumed deflection function *)
  phi = Expand [ (x^2 - a^2/4) ^2 * (y^2 - b^2/4)
^2];
  w = a00 *phi;
(* Evaluate the Galerkin integral *)
  GalInt = Simplify [Integrate [ (Dx*D [w, {x, 4} ] +
  2*H*D [w, {x, 2}, {y, 2}] +
  Dy*D [w, {y, 4} ] - q0) *phi,
  {x, - a/2, a/2},{y, - b/2, b/2}]];
(* Solve the Galerkin equation for a00 *)
  Galsol = Simplify [Solve [GalInt = = 0, a00 ] ];
(* Evaluate the plate response *)
```

```
w = Simplify [w /. Galsol /.b - > a*c]
d2x = D [w, {x, 2}]; d2y = D [w, {y, 2}]; d12 = D [w, x, y];
Mx = Simplify [ - (Dx*d2x + D1*d2y) ]
My = Simplify [ - (D1*d2x + Dy*d2y) ]
Mxy = Simplify [ - 2*Dxy*d12]
Qx = Simplify [ - (Dx*D [d2x, x] + H*D [d2y, x]) ]
Qy = Simplify [ - (H*D [d2x, y] + Dy*D [d2y, y]) ]
```

Appendix B. Mathematica procedure for the second approximation

```
(* Specify the assumed deflection function *)
phifact = Expand [ (x^2 - a^2/4) ^2* (y^2 - b^2/4) ^2];
phi [1] = phifact;
phi [2] = phifact*x^2;
phi [3] = phifact *y^2;
w = Sum [a[i] *phi [i], {i, 3} ];
(* Evaluate the Galerkin integral *)
GalInt = Simplify [Table [Integrate [ (Dx*D [w, {x, 4}] +
2*H*D [w, {x, 2}, {y, 2}] +
Dy*D [w, {y, 4}] -q0) *phi [i],
{x, -a/2, a/2}, {y, -b/2, b/2}], {i, 3}]];
(* Solve the Galerkin equation for a_i (i = 1, 2, 3) = (a_00, a_02, a_20) *)
Galsol = Simplify [Solve [GalInt = = 0, Table [a[i], {i, 3} ] ] ];
(* Evaluate the plate response as in Appendix A procedure *)
```

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