Exact Method for Static and Natural Vibration Analyses of Bi-Periodic Structures

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Abstract: The U-transformation technique has been applied successfully to the analysis of periodic structures and nearly periodic structures. In this study the technique will be extended to the analysis of bi-periodic structures under static loading or natural vibration, since it is possible to uncouple the governing equation by applying the U-transformation twice. To explain the method used in this paper, a simple cyclic system with bi-periodicity is considered first. It helps to demonstrate the procedures for uncoupling the static equilibrium equation to obtain the closed form solution for displacement and the natural vibration equation to obtain the natural frequencies and modes. Then a continuous beam with equidistant rigid and elastic supports (a structure with bi-periodicity), subjected to a concentrated load, is studied and the generalized analytical solution is derived. Some numerical results are also given. Though not illustrated, it is obvious that the arbitrary static loading condition can be dealt with in the same manner.

INTRODUCTION

The earliest study on bi-periodic structures might be the analysis of compound periodic structures by Lin and McDaniel (1969) where the transfer matrix method was used. The wave propagation in doubly periodic structures was investigated, using a wave approach by Sen Gupta (1972). Mead (1975) had considered wave propagation in bi-periodic structures of monocoupled and multicoupled systems. The dynamics of bi-periodic structures was studied by McDaniel and Carroll (1982) where the analysis was formulated from standard stiffness and transmission methods. The research works presented in the foregoing text are concerned with the wave propagation and natural vibration for bi-periodic structures and are not applicable to the static and forced vibration analyses.

The U-transformation technique developed by Cai et al. (1988) has been applied to the static and forced vibration analyses of periodic structures (Cheung et al. 1989; Cai et al. 1989). Recently the U-transformation is used to analyze nearly periodic structures by Cai et al. (1995). It is conceived that bi-periodic structures may be treated as the nearly periodic ones.

The static and natural vibration analyses of bi-periodic structures are considered in this study. The U-transformation will be used to uncouple the governing equation of structures with bi-periodicity. As a result a set of simultaneous equations with single periodicity will be obtained. Then by applying the U-transformation again, it will lead to the analytical solution. To explain this method a simple cyclic system with bi-periodicity is considered first. Then a continuous beam with equidistant rigid and elastic supports, subjected to a concentrated load, is analyzed. The closed form solution is derived.

UNCOUPLING OF GOVERNING EQUATION FOR ROTATIONALLY BI-PERIODIC STRUCTURES

The method to be used will be demonstrated by using a simple model. At the outset, a structure with cyclic bi-periodicity is considered. A general model is illustrated in Fig. 1

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Note. Associate Editor: Dimitrios Karamanlidis. Discussion open until January 1, 1999. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on February 7, 1997. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 124, No. 8, August, 1998. ©ASCE, ISSN 0733-9399/98/0008-0836–0841/\$8.00 + \$.50 per page. Paper No. 15093.

where all the coupling springs are of the same stiffness k. K and $K + \Delta K$ denote the stiffness for two kinds of cantilever beams, M and $M + \Delta M$ denote the lumped masses, and F_j, x_j denote the load-displacement for the jth subsystem.

Static Problem

The equilibrium equations can be expressed as

$$(K + 2k)x_j - k(x_{j+1} + x_{j-1}) = -\Delta K x_j + F_j, \quad j = p, 2p, \ldots, np \tag{1a}$$

$$(K + 2k)x_j - k(x_{j+1} + x_{j-1}) = F_j, \quad j \neq p, 2p, \ldots, np \tag{1b}$$

where $p, 2p, \ldots, np$ = ordinal numbers of the subsystems with stiffness $K + \Delta K$. The term $-\Delta K x_j$ on the right-hand side of (1a) may be treated as the load as well as F_j.

One can now apply the U-transformation (Cai et al. 1988) to (1a) and (1b). The U- and inverse U-transformation may be defined as

$$x_j = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} e^{i(j-1)r\psi} q_r, \quad j = 1, 2, \ldots, N \tag{2a}$$

and

$$q_r = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i(j-1)r\psi} x_j, \quad r = 1, 2, \ldots, N \tag{2b}$$

in which $\psi = 2\pi/N$; $i = \sqrt{-1}$; and N = total number of subsystems, i.e., $N = pn$.

The equilibrium equations (1a) and (1b) may be expressed in terms of the generalized displacements q_r ($r = 1, 2, \ldots, N$) as

$$(K + 2k)q_r - 2k \cos r\psi q_r = f_r + f_r^*, \quad r = 1, 2, \ldots, N \tag{3}$$

where

$$f_r = -\frac{\Delta K}{\sqrt{N}} \sum_{m=1}^{n} e^{-i(mp-1)r\psi} x_{mp} \tag{4}$$

$$f_r^* = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i(j-1)r\psi} F_j \tag{5}$$

The generalized displacement q_r in (3) may be formally expressed as

$$q_r = \frac{f_r + f_r^*}{K + 2k(1 - \cos r\psi)} \tag{6}$$

Substituting (6) and (4) into (2a) yields

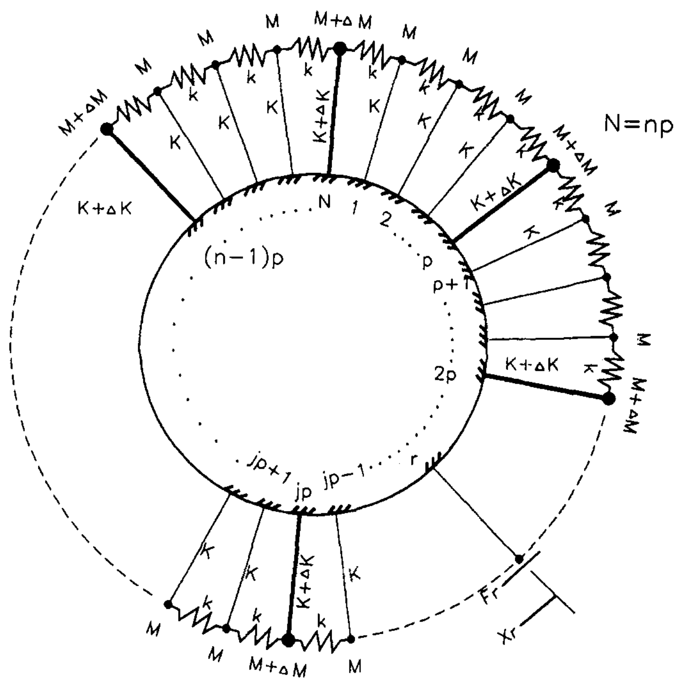

FIG. 1. Rotationally Bi-Periodic System

$$x_j = -\frac{\Delta K}{N}\sum_{m=1}^{n}\sum_{r=1}^{N}\frac{e^{i(j-mp)r\psi}}{K + 2k(1-\cos r\psi)}x_{mp} + x_j^*,$$

$$j = 1, 2, \ldots, N \qquad (7)$$

in which

$$x_j^* = \frac{1}{\sqrt{N}}\sum_{r=1}^{N}\left[e^{i(j-1)r\psi}\frac{f_r^*}{K + 2k(1-\cos r\psi)}\right] \qquad (8)$$

x_j^* ($j = 1, 2, \ldots, N$) = solution for the perfect periodic system (i.e., $\Delta K = 0$) subjected to the same loading as that acting on the bi-periodic system. When the specific loading condition is given, x_j^* ($j = 1, 2, \ldots, N$) can be obtained from (8) and (5).

Inserting $j = sp$ ($s = 1, 2, \ldots, n$) in (7) gives

$$X_s = -\Delta K\sum_{m=1}^{n}\beta_{s,m}X_m + X_s^*, \quad s = 1, 2, \ldots, n \qquad (9)$$

where

$$X_s \equiv x_{sp}; \quad X_s^* \equiv x_{sp}^* \qquad (10a,b)$$

and

$$\beta_{s,m} = \frac{1}{N}\sum_{r=1}^{N}\frac{e^{i(s-m)pr\psi}}{K + 2k(1-\cos r\psi)}, \quad s, m = 1, 2, \ldots, n \qquad (11)$$

$\beta_{s,m}$ = influence coefficient for the perfect periodic system.

By using the U-transformation once, the equilibrium equations (1a) and (1b) with N (=pn) unknowns become (9) with n unknown. Noting that $\beta_{s,m}$ ($s, m = 1, 2, \ldots, n$) shown in (11) have the cyclic periodicity, i.e.

$$\beta_{1,1} = \beta_{2,2} = \cdots = \beta_{n,n} \qquad (12a)$$

$$\beta_{s,1} = \beta_{s+1,2} = \cdots = \beta_{n,n-s+1} = \beta_{1,n-s+2} = \cdots = \beta_{s-1,n},$$

$$s = 2, 3, \ldots, n \qquad (12b)$$

One can now apply the U-transformation again to (9), i.e., introducing

$$X_s = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}e^{i(s-1)j\varphi}Q_j, \quad s = 1, 2, \ldots, n \qquad (13a)$$

or

$$Q_j = \frac{1}{\sqrt{n}}\sum_{s=1}^{n}e^{-i(s-1)j\varphi}X_s, \quad j = 1, 2, \ldots, n \qquad (13b)$$

with $\varphi = 2\pi/n$, (9) becomes

$$Q_j = -\Delta K\sum_{m=1}^{n}\beta_{m1}e^{-i(m-1)j\varphi}Q_j + b_j, \quad j = 1, 2, \ldots, n \qquad (14)$$

where

$$b_j = \frac{1}{\sqrt{n}}\sum_{s=1}^{n}e^{-i(s-1)j\varphi}X_s^* \qquad (15)$$

Obviously the solution for Q_j of (14) is

$$Q_j = \frac{b_j}{1 + \Delta K\sum_{m=1}^{n}\beta_{m1}e^{-i(m-1)j\varphi}} \qquad (16)$$

Substituting (11) into (16) results in

$$Q_j = \frac{b_j}{1 + \frac{\Delta K}{p}\sum_{r=1}^{p}[K + 2k - 2k\cos[j + (r-1)n]\psi]^{-1}} \qquad (17)$$

Inserting (17) in the later U-transformation (13a) yields

$$x_{sp} \equiv X_s = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}e^{i(s-1)j\varphi}b_j$$

$$\Big/\left\{1 + \frac{\Delta K}{p}\sum_{r=1}^{p}[K + 2k - 2k\cos[j + (r-1)n]\psi]^{-1}\right\},$$

$$s = 1, 2, \ldots, n \qquad (18)$$

in which

$$\varphi = 2\pi/n; \quad \psi = 2\pi/N; \quad N = pn \qquad (19a\text{–}c)$$

The exact solution for x_j ($j = 1, 2, \ldots, N$) of (1a) and (1b) can be found by inserting (18) into (7). The exact solution includes two independent parameters of periodicity for bi-periodic systems, i.e., p and n. If p, n, and loads are given, the displacement for each subsystem can be calculated.

For example, consider the following case:

$$p = 3,\ n = 2 \text{ (as a result } N = 6,\ \varphi = \pi \text{ and } \psi = \pi/3) \qquad (20a)$$

$$F_3 = F_6 = P \quad \text{and} \quad F_j = 0, \quad j \neq 3, 6 \qquad (20b)$$

Inserting (20b) into (5) yields

$$f_r^* = \frac{2P}{\sqrt{6}}e^{-i2\pi r/3}, \quad r = 2, 4, 6 \qquad (21a)$$

$$f_r^* = 0, \quad r = 1, 3, 5 \qquad (21b)$$

and then substituting (20a) and (21) into (8) gives

$$x_j^* = \frac{P}{3}\left[\frac{2\cos(2j\pi/3)}{K + 3k} + \frac{1}{K}\right]$$

that is

$$x_j^* = \frac{Pk}{K(K + 3k)}, \quad j = 1, 2, 4, 5 \qquad (22a)$$

$$x_j^* = \frac{P(K + k)}{K(K + 3k)}, \quad j = 3, 6 \qquad (22b)$$

It can be verified that x_j^* shown in (22) is the displacement solution for the system with $\Delta K = 0$ subjected to the loads shown in (20b).

Noting the definition shown in (10) and $p = 3$ gives

$$X_1^* = X_2^* = \frac{P(K + k)}{K(K + 3k)} \quad (23)$$

and then substituting the preceding equation and $\varphi = \pi$ into (15) yields

$$b_1 = 0; \quad b_2 = \frac{\sqrt{2}P(K + k)}{K(K + 3k)} \quad (24a,b)$$

Finally, substituting (24) and (20a) into (18) results in

$$x_3 = x_6 = \frac{P(K + k)}{K(K + 3k) + \Delta K(K + k)} \quad (25)$$

then inserting (25) and (22) into (7) results in

$$x_1 = x_2 = x_4 = x_5 = \frac{Pk}{K(K + 3k) + \Delta K(K + k)} \quad (26)$$

The displacement x_j ($j = 1, 2, \ldots, 6$) shown in (25) and (26) satisfies the equilibrium equations (1a) and (1b) with $p = 3$ and $n = 2$.

Natural Vibration

The natural vibration equation for the bi-periodic system shown in Fig. 1 may be expressed as

$$(K + 2k - M\omega^2)x_j - k(x_{j+1} + x_{j-1}) = -(\Delta K - \Delta M\omega^2)x_j,$$
$$j = p, 2p, \ldots, np \quad (27a)$$

$$(K + 2k - M\omega^2)x_j - k(x_{j+1} + x_{j-1}) = 0, \quad j \neq p, 2p, \ldots, np \quad (27b)$$

where ω = natural frequency; and x_j = amplitude for jth subsystem. Applying the U-transformation (2) to (27a) and (27b) results in

$$(K + 2k - M\omega^2)q_r - 2k\cos r\psi q_r = f_r, \quad r = 1, 2, \ldots, N \quad (28)$$

where

$$f_r = -\frac{(\Delta K - \Delta M\omega^2)}{\sqrt{N}} \sum_{m=1}^{n} e^{-i(mp-1)r\psi} x_{mp} \quad (29)$$

and then

$$q_r = \frac{f_r}{K + 2k - 2k\cos r\psi - M\omega^2} \quad (30)$$

Substituting (30) and (29) into (2a) yields

$$x_j = -\frac{(\Delta K - \Delta M\omega^2)}{N} \sum_{m=1}^{n} \sum_{r=1}^{N} \frac{e^{i(j-mp)r\psi}}{K + 2k - M\omega^2 - 2k\cos r\psi} x_{mp},$$
$$j = 1, 2, \ldots, N \quad (31)$$

Introducing the notation $X_s \equiv x_{sp}$ and inserting $j = sp$ ($s = 1, 2, \ldots, n$) in (31) gives

$$X_s = -(\Delta K - \Delta M\omega^2) \sum_{m=1}^{n} \beta_{sm}^* X_m, \quad s = 1, 2, \ldots, n \quad (32)$$

where

$$\beta_{sm}^* = \frac{1}{N} \sum_{r=1}^{N} \frac{e^{i(s-m)pr\psi}}{K + 2k - M\omega^2 - 2k\cos r\psi}, \quad s, m = 1, 2, \ldots, n \quad (33)$$

β_{sm}^* = harmonic influence coefficient for the considered system with $\Delta K = \Delta M = 0$.

Applying the U-transformation (13a) and (13b) to (32) results in

$$Q_j = -(\Delta K - \Delta M\omega^2) \sum_{m=1}^{n} \beta_{m1}^* e^{-i(m-1)j\varphi} Q_j, \quad j = 1, 2, \ldots, n \quad (34)$$

When x_{sp} ($s = 1, 2, \ldots, n$) are not vanishing, the frequency equation may be expressed as

$$1 + (\Delta K - \Delta M\omega^2) \sum_{m=1}^{n} \beta_{m1}^* e^{-i(m-1)j\varphi} = 0, \quad j = 1, 2, \ldots, n \quad (35)$$

Substituting (33) and $\varphi = p\psi$ into (35), the frequency equation becomes

$$1 + (\Delta K - \Delta M\omega^2)\frac{1}{p} \sum_{r=1}^{p} \{K + 2k - M\omega^2$$
$$- 2k\cos[j + (r - 1)n]\psi\}^{-1} = 0, \quad j = 1, 2, \ldots, n \quad (36a)$$

When x_{sp} ($s = 1, 2, \ldots, n$) are vanishing, the frequency equation can be obtained from (28) and (29) as

$$K + 2k(1 - \cos r\psi) - M\omega^2 = 0 \quad (36b)$$

where the half-wave number $2r$ must be equal to integer times n and $<2N$.

Consider now the case shown in (20a) and $M = 0$, $\Delta M = M_0$. For this case, the frequency equation (36a) becomes

$$1 + \frac{\Delta K - M_0\omega^2}{3} \sum_{r=1}^{3} \left\{K + 2k - 2k\cos[j + 2(r - 1)]\frac{\pi}{3}\right\}^{-1} = 0,$$
$$j = 1, 2 \quad (37)$$

The solution for ω^2 of (37) may be expressed as

$$\omega^2 = \frac{\Delta K}{M_0} + \frac{1}{M_0 I_j}, \quad j = 1, 2 \quad (38)$$

where

$$I_j = \frac{1}{3} \sum_{r=1}^{3} \left\{K + 2k - 2k\cos[j + 2(r - 1)]\frac{\pi}{3}\right\}^{-1}, \quad j = 1, 2 \quad (39a)$$

or

$$I_1 = \frac{K + 3k}{(K + k)(K + 4k)}; \quad I_2 = \frac{K + k}{K(K + 3k)} \quad (39b,c)$$

Inserting (39b,c) into (38) gives

$$\omega_1^2 = \frac{\Delta K}{M_0} + \frac{K(K + 3k)}{M_0(K + k)}; \quad \omega_2^2 = \frac{\Delta K}{M_0} + \frac{(K + k)(K + 4k)}{M_0(K + 3k)} \quad (40a,b)$$

where ω_1 and ω_2 = first and second natural frequencies, respectively.

Consider now the natural modes. The generalized displacements for the first natural mode may be expressed as

$$Q_2 \neq 0; \quad Q_1 = 0 \quad (41a,b)$$

Substituting (41) into (13a) with $n = 2$ and $\varphi = \pi$ gives

$$x_3 \equiv X_1 = A; \quad x_6 \equiv X_2 = A \quad (42a,b)$$

where A = arbitrary constant. Substituting (42) and $\omega^2 = \omega_1^2$ shown in (40a) into (31) results in

$$x_1 = x_2 = x_4 = x_5 = \frac{k}{K + k}A \quad (43)$$

The first mode is obtained as shown in (42) and (43).

In a manner similar to that for finding the first natural mode, the second natural mode ($Q_1 \neq 0$, $Q_2 = 0$) may be found as

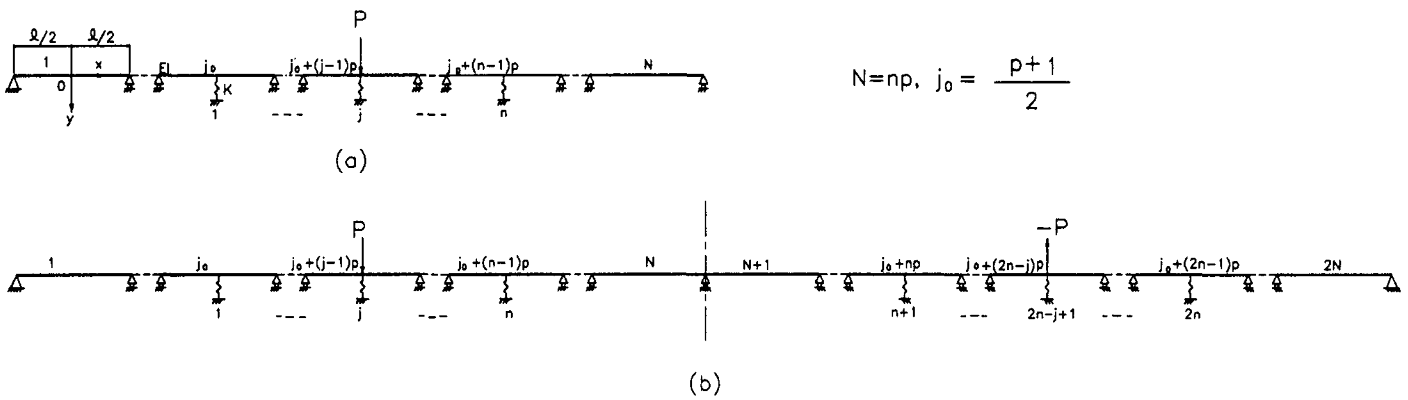

FIG. 2. Continuous Beam with Bi-Periodic Supports: (a) Actual System; (b) Equivalent System

$$x_3 \equiv X_1 = A; \quad x_6 \equiv X_2 = -A \qquad (44a,b)$$

and

$$x_1 = x_5 = -\frac{k}{K + 3k}A; \quad x_2 = x_4 = \frac{k}{K + 3k}A \qquad (44c,d)$$

The natural frequencies and modes shown in (40) and (42)–(44) are in agreement with those obtained from the usual stiffness method.

The preceding described method can be extended to analyzing the forced vibration of bi-periodic systems subjected to harmonic loading.

STATIC ANALYSIS OF CONTINUOUS BEAMS WITH TWO KINDS OF PERIODIC SUPPORTS

Consider a beam with uniform flexural rigidity EI running over $N + 1$ number of roller supports and n elastic supports as shown in Fig. 2(a), where K denotes the stiffness of the elastic supports and l denotes the span length between any two adjacent roller supports. The distance between any two adjacent elastic supports is pl. It is assumed that each elastic support is located at midspan and $N = pn$, and a symmetric plane of the beam exists, i.e., p must be an odd number.

To form an equivalent structure with cyclic bi-periodicity for the beam considered, it is necessary to extend the original beam by its symmetrical image and apply the antisymmetric loading on the corresponding extended part as shown in Fig. 2(b). Such an equivalent system can be regarded as a cyclic bi-periodic system, because the slopes and moments at both extreme ends are the same. The simply supported boundary conditions at both extreme ends for the original beam can be satisfied automatically in its equivalent system.

In each span a local coordinate system oxy with the origin 0 at its midspan is established. The equilibrium equations may be expressed as

$$EI\frac{d^4w_j(x)}{dx^4} = F_j(x) - Kw_j(0)\delta(x),$$

$$j = j_0, j_0 + p, \ldots, j_0 + (2n - 1)p \qquad (45a)$$

$$EI\frac{d^4w_j(x)}{dx^4} = F_j(x), \quad j \neq j_0, j_0 + p, \ldots, j_0 + (2n - 1)p \qquad (45b)$$

where $j_0 = (p + 1)/2$; and j_0 = ordinal number of the span with the first elastic support; $w_j(x)$ and $F_j(x)$ = deflection and loading functions for jth span; $\delta(x)$ = Dirac delta function.

The loading functions must satisfy the following antisymmetric condition:

$$F_{2N-j+1}(x) = -F_j(-x), \quad j = 1, 2, \ldots, N \qquad (46)$$

in which $F_j(x)$ ($j = 1, 2, \ldots, N$) = real loading acting on the original beam.

Continuity across the roller supports requires the following conditions to be satisfied:

$$w_j\left(-\frac{l}{2}\right) = 0; \quad w_j\left(\frac{l}{2}\right) = 0 \qquad (47a,b)$$

$$w_j'\left(\frac{l}{2}\right) = w_{j+1}'\left(-\frac{l}{2}\right); \quad w_j''\left(\frac{l}{2}\right) = w_{j+1}''\left(-\frac{l}{2}\right) \qquad (47c,d)$$

where $j = 1, 2, \ldots, 2N$; and a prime denotes differentiation with respect to x and $w_{2N+1} \equiv w_1$ due to the cyclic periodicity.

Introducing the U-transformation

$$w_j(x) = \frac{1}{\sqrt{2N}}\sum_{r=1}^{2N} e^{i(j-1)r\psi}q_r(x), \quad j = 1, 2, \ldots, 2N \qquad (48a)$$

$$q_r(x) = \frac{1}{\sqrt{2N}}\sum_{j=1}^{2N} e^{-i(j-1)r\psi}w_j(x), \quad r = 1, 2, \ldots, 2N \qquad (48b)$$

with $\psi = \pi/N$ into (45a), (45b), and (47) results in

$$EI\frac{d^4q_r(x)}{dx^4} = f_r(x) + f_r^*(x), \quad r = 1, 2, \ldots, 2N \qquad (49)$$

and

$$q_r\left(\frac{l}{2}\right) = 0; \quad q_r\left(-\frac{l}{2}\right) = 0 \qquad (50a,b)$$

$$q_r'\left(\frac{l}{2}\right) = e^{ir\psi}q_r'\left(-\frac{l}{2}\right); \quad q_r''\left(\frac{l}{2}\right) = e^{ir\psi}q_r''\left(-\frac{l}{2}\right) \qquad (50c,d)$$

where

$$f_r = -\frac{K\delta(x)}{\sqrt{2N}}\sum_{m=1}^{2n} e^{-i[j_0+(m-1)p-1]r\psi}w_{j_0+(m-1)p}(0) \qquad (51)$$

$$f_r^* = \frac{1}{\sqrt{2N}}\sum_{j=1}^{2N} e^{-i(j-1)r\psi}F_j(x) \qquad (52)$$

If the loading condition is given, the generalized load f_r^* can be found. The formal solution for q_r of (49) subject to boundary condition (50) may be expressed as

$$q_r(x) = q_r^0(x) + q_r^*(x) \qquad (53)$$

where

$$q_r^0(x) = -\frac{K}{EI\sqrt{2N}}\sum_{m=1}^{2n} e^{-i[j_0+(m-1)p-1]r\psi}w_{j_0+(m-1)p}(0)\left(C_{r0} + C_{r1}x + C_{r2}x^2 + C_{r3}x^3 + \frac{1}{12}|x|^3\right), \quad r = 1, 2, \ldots, 2N \qquad (54)$$

$$C_{r0} = \frac{l^3}{384}\left(\frac{7 - \cos r\psi}{2 + \cos r\psi}\right); \quad C_{r1} = -i\frac{l^2}{64}\left(\frac{\sin r\psi}{2 + \cos r\psi}\right) \qquad (55a,b)$$

$$C_{r2} = -\frac{l}{32}\left(\frac{5 + \cos r\psi}{2 + \cos r\psi}\right); \quad C_{r3} = \frac{i}{16}\left(\frac{\sin r\psi}{2 + \cos r\psi}\right) \quad (55c,d)$$

and $q_r^*(x)$ = generalized displacement for the cyclic periodic system with $K = 0$ subjected to the same loading as that acting on the original system.

Substituting (53)–(55) into (48a) yields

$$w_j(x) = w_j^0(x) + w_j^*(x) \quad (56)$$

in which

$$w_j^0(x) = -\frac{K}{2EIN}\sum_{m=1}^{2n}\sum_{r=1}^{2N} e^{i[j-j_0-(m-1)p]r\psi} w_{j_0+(m-1)p}(0)\left(C_{r0} + C_{r1}x + C_{r2}x^2 + C_{r3}x^3 + \frac{1}{12}|x|^3\right), \quad j = 1, 2, \ldots, 2N \quad (57a)$$

$$w_j^*(x) = \frac{1}{\sqrt{2N}}\sum_{r=1}^{2N} e^{i(j-1)r\psi} q_r^*(x), \quad j = 1, 2, \ldots, 2N \quad (57b)$$

$w_j^*(x)$ = deflection function of jth span for the periodic system with K vanishing under the same loading as that acting on the original system.

Inserting $j = j_0 + (s - 1)p$ and $x = 0$ into (56) and (57a) gives

$$W_s = -\frac{K}{EI}\sum_{m=1}^{2n} \beta_{sm} W_m + W_s^*, \quad s = 1, 2, \ldots, 2n \quad (58)$$

where

$$\beta_{sm} = \frac{1}{2N}\sum_{r=1}^{2N} e^{i(s-m)pr\psi} C_{r0}, \quad s, m = 1, 2, \ldots, 2n \quad (59)$$

$$W_s = w_{j_0+(s-1)p}(0), \; W_s^* = w^*_{j_0+(s-1)p}(0), \quad s = 1, 2, \ldots, 2n \quad (60)$$

It is obvious that β_{sm} ($s, m = 1, 2, \ldots, 2n$) shown in (59) satisfy the cyclic periodicity condition as shown in (12), where n should be replaced by $2n$. To solve the simultaneous equation (58), it is convenient to introduce the U-transformation as illustrated next.

Let

$$W_s = \frac{1}{\sqrt{2n}}\sum_{j=1}^{2n} e^{i(s-1)j\varphi} Q_j, \quad s = 1, 2, \ldots, 2n \quad (61a)$$

or

$$Q_j = \frac{1}{\sqrt{2n}}\sum_{s=1}^{2n} e^{-i(s-1)j\varphi} W_s, \quad j = 1, 2, \ldots, 2n \quad (61b)$$

where $\varphi = \pi/n$.

Equation (58) can be expressed in terms of Q_j ($j = 1, 2, \ldots, 2n$) as

$$Q_j = -\frac{K}{EI}\sum_{m=1}^{2n} \beta_{m1} e^{-i(m-1)j\varphi} Q_j + b_j, \quad j = 1, 2, \ldots, 2n \quad (62)$$

in which

$$\beta_{m1} = \frac{1}{2N}\sum_{r=1}^{2N} e^{i(m-1)pr\psi} C_{r,0}, \quad m = 1, 2, \ldots, 2n \quad (63)$$

$$b_j = \frac{1}{\sqrt{2n}}\sum_{s=1}^{2n} e^{-i(s-1)j\varphi} W_s^*, \quad j = 1, 2, \ldots, 2n \quad (64)$$

Substituting (63) into (62) results in

$$Q_j = -\frac{K}{EI}\frac{1}{p}\sum_{r=1}^{p} C_{j+(r-1)2n,0} Q_j + b_j \quad (65)$$

where $C_{j+(r-1)2n,0}$ has been defined as (55), i.e.

$$C_{j+(r-1)2n,0} = \frac{l^3}{384}\left(\frac{7 - \cos[j + (r - 1)2n]\psi}{2 + \cos[j + (r - 1)2n]\psi}\right) \quad (66)$$

The solution for Q_j of (65) can be written as

$$Q_j = b_j \bigg/ \left\{1 + \frac{Kl^3}{384EI}\frac{1}{p}\sum_{r=1}^{p} \frac{7 - \cos[j + (r - 1)2n]\psi}{2 + \cos[j + (r - 1)2n]\psi}\right\}, \quad j = 1, 2, \ldots, 2n \quad (67)$$

In which b_j shown in (64) is dependent on the loading condition. When the specific load is given, b_j can be found without difficulty.

Consider a concentrated load of magnitude P acting at the midpoint of the middle span, say kth span [i.e., $k = (N + 1)/2$ and N is an odd number], as shown in Fig. 2(a) where $j = k$. For the equivalent system shown in Fig. 2(b), equal but opposite concentrated loads must be applied to the kth and $(2N - k + 1)$th spans with all other spans unloaded, i.e.

$$F_k(x) = P\delta(x); \quad F_{2N-k+1}(x) = -P\delta(x); \quad F_j(x) = 0 \quad (68a–c)$$

where $j \neq k$, $2n - k + 1$; $k = (N + 1)/2$; and N = odd number.

Inserting (68) into (52) yields

$$f_r^*(x) = \frac{2P\delta(x)}{\sqrt{2N}} e^{-i(k-1)r\psi}, \quad r = 1, 3, \ldots, 2N - 1 \quad (69a)$$

$$f_r^*(x) = 0, \quad r = 2, 4, \ldots, 2N \quad (69b)$$

and $\psi = \pi/N$.

Noting that $q_r^*(x)$ represents the solution for $q_r(x)$ of (49) with $f_r(x)$ vanishing subject to boundary condition (50) and $f_r^*(x)$ is shown in (69), $q_r^*(x)$ can be found as

$$q_r^*(x) = \frac{2Pe^{-i(k-1)r\psi}}{EI\sqrt{2N}}\left(C_{r0} + C_{r1}x + C_{r2}x^2 + C_{r3}x^3 + \frac{1}{12}|x|^3\right), \quad r = 1, 3, \ldots, 2N - 1 \quad (70a)$$

$$q_r^*(x) = 0, \quad r = 2, 4, \ldots, 2N \quad (70b)$$

where C_{r0}–C_{r3} have the same definition as those shown in (55).

Substituting (70) into (57b) yields

$$w_j^*(x) = \frac{P}{EIN}\sum_{r=1,3,5}^{2N-1} e^{i(j-k)r\psi}\left(C_{r0} + C_{r1}x + C_{r2}x^2 + C_{r3}x^3 + \frac{1}{12}|x|^3\right), \quad j = 1, 2, \ldots, 2N \quad (71)$$

Noting $W_s^* = w^*_{j_0+(s-1)p}(0)$, $k = (N + 1)/2$, and $j_0 = (p + 1)/2$, inserting $j = j_0 + (s - 1)p$ and $x = 0$ into (71) gives

$$W_s^* = \frac{P}{EIN}\sum_{r=1,3,5}^{2N-1} e^{-ir\pi/2} e^{i(s-1/2)pr\psi} C_{r0} \quad (72)$$

and then substituting (72) into (64) results in

$$b_j = \frac{2P}{EIp}\frac{1}{\sqrt{2n}} e^{ij(\varphi-\pi)/2}\sum_{r=1}^{p} C_{j+(r-1)2n,0}, \quad j = 1, 3, \ldots, 2n - 1 \quad (73a)$$

$$b_j = 0, \quad j = 2, 4, \ldots, 2n \quad (73b)$$

Since N, p are odd numbers, therefore, n ($=N/p$) is also an odd number. This property has been used for deriving (73a) and (73b).

Afterward inserting (73a), (73b), and (66) into (67) yields

$$Q_j = \frac{Pl^3}{192EI}\frac{1}{\sqrt{2n}} e^{ij(\varphi-\pi)/2} \sum_j \bigg/ \left(1 + K_0 \sum_j\right), \quad j = 1, 3, \ldots, 2n - 1 \quad (74a)$$

TABLE 1. Maximum Deflections of Continuous Beams with Bi-Periodic Supports Subjected to Concentrated Load P at Midpoint ($p = 3$)

K_0 (1)	$N(n)$ 3 (1) (2)	9 (3) (3)	15 (5) (4)	21 (7) (5)
0.0	4.40000	4.19623	4.19615	4.19615
0.1	3.05556	2.95515	2.95514	2.95514
0.2	2.34043	2.28083	2.28083	2.28083
0.5	1.37500	1.35410	1.35410	1.35410
1.0	0.81481	0.80741	0.80741	0.80741
2.0	0.44898	0.44672	0.44672	0.44672
Multiplier	$Pl^3/384EI$	$Pl^3/384EI$	$Pl^3/348EI$	$Pl^3/348EI$

$$Q_j = 0, \quad j = 2, 4, \ldots, 2n \tag{74b}$$

where

$$\sum_j = \frac{1}{p}\sum_{r=1}^{p}\frac{7 - \cos[j + (r - 1)2n]\psi}{2 + \cos[j + (r - 1)2n]\psi} \tag{75}$$

$$K_0 = \frac{Kl^3}{384EI}; \quad \psi = \frac{\pi}{N} \tag{76a,b}$$

K_0 = nondimensional parameter of the stiffness for the elastic support.

Substituting (74a) and (74b) into (61a) yields

$$w_{j_0+(s-1)p}(0) \equiv W_s = \frac{Pl^3}{384EIn}\sum_{j=1,3}^{2n-1}\left[e^{i[s-(n+1)/2]j\varphi}\sum_j \Big/ \left(1 + K_0\sum_j\right)\right], \quad s = 1, 2, \ldots, 2n \tag{77}$$

and $\varphi = \pi/n$.

Now the deflection function for each span can be found by inserting (71), (57a), and (77) into (56). The maximum deflection occurs at the midpoint of the loaded span. The maximum deflection can be obtained by inserting $s = (n + 1)/2$ in (77) as

$$w_{max} = w_{(N+1)/2}(0) = \frac{Pl^3}{384EI}\frac{1}{n}\sum_{j=1,3}^{2n-1}\left[\sum_j \Big/ \left(1 + K_0\sum_j\right)\right] \tag{78}$$

in which Σ_j shown in (75) is dependent on p and N. Noting $N = np$, (78) includes two independent parameters n or N and p besides K_0. Some numerical results for (78) are given in Table 1 where $p = 3$ and the total number of spans N and the nondimensional stiffness K_0 take several values, respectively.

Consider the particular case of $K = 0$. By substituting $K_0 = 0$ and (75) into (78), the maximum deflection can be expressed as

$$w_{max} = \frac{Pl^3}{384EI}\frac{1}{np}\sum_{j=1,3}^{2n-1}\sum_{r=1}^{p}\frac{7 - \cos[j + (r - 1)2n]\psi}{2 + \cos[j + (r - 1)2n]\psi} = \frac{Pl^3}{384EI}\frac{1}{N}\sum_{m=1,3}^{2N-1}\frac{7 - \cos m\psi}{2 + \cos m\psi} \tag{79}$$

The preceding result is in agreement with that obtained by Cheung et al. (1989).

CONCLUSIONS

In this paper it has been demonstrated that the U-transformation technique can be applied to the static analysis of continuous beams with periodic rigid and elastic supports, and the rigorous analytical solution has been derived for the concentrated load condition. It is obvious that the present method of solution is applicable to an arbitrary static loading condition.

This method may be extended to the analysis of two-dimensional structures with bi-periodicity. For example, rectangular stiffened plates (with periodic roller supports and two opposite continuous simply supported edges) and simply supported rectangular plates (with two types of periodic ribs) fall within the solvable two dimensional bi-periodic structures, but the derivation of the solutions will be lengthy.

The U-transformation technique may also be applied to the natural vibration analysis of some bi-periodic structures as shown in the section entitled Uncoupling of Governing Equation for Rotationally Bi-Periodic Structures. Generally, the frequency equation may be derived, but the solution for natural frequency cannot be obtained explicitly.

APPENDIX. REFERENCES

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