

ITERATIVE COUPLING OF BEM AND FEM FOR THE SOLUTION OF ELASTO-PLASTIC FRACTURE MECHANICS PROBLEMS

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In this paper we extend the application of the sequential Dirichlet-Neumann iterative boundary element-finite element coupling method to elasto-plasticity. The successive computation of the displacements and forces/tractions on the interface of the finite element and boundary element sub-domains is performed through an iterative procedure. The procedure is implemented in a computer program and is tested through linear elastic fracture mechanics and elasto-plastic fracture mechanics problems.

Keywords: Boundary Element Method; Finite Element Method; Elasto-Plasticity; Fracture Mechanics, Iterative Methods; Coupling.

1 Introduction

For certain categories of problems, neither the boundary element method (BEM) nor the finite element method (FEM) is best suited and it is natural to attempt to couple these two methods in an effort to create a finite element-boundary element method (FEBEM) that combines all their advantages and reduces their disadvantages.

Unfortunately, the systems of equations, produced by the two methods, are expressed in terms of different variables and cannot be linked as they stand. The coupling of the two methods has been a topic of great interest for more than two decades. The conventional coupling methods [1-17] employ an entire unified equation for the whole domain, by combining the discretized equations for the BEM and FEM sub-domains. The algorithm for constructing an entire equation is highly complicated when compared with that for each single equation. In order to overcome the stated inconvenience, iterative domain decomposition coupling approaches were developed [18-24], where there is a no need to combine the coefficient matrix for the FEM and BEM sub-domains. A second advantage is that different formulation of the FEM and BEM can be adopted as base programs for coupling the computer codes only. In these coupling algorithms, separate computing for each sub-domain and successive renewal of the variables on the interface of the both sub-domains are performed to reach the final convergence. Gerstle et al. [18] and Perera et al. [19] presented solution schemes, which utilize the conjugate gradient method and the Schur complement, respectively, for the renewal of the unknowns at the interface. Kamiya et al. [20] employed the renewal schemes known as Schwarz Neumann-Neumann and Schwarz Dirichlet-Neumann. Kamiya and Iwase [21] introduced an iterative analysis using conjugate gradient and condensation. Lin et al. [22], and Feng and Owen [23] presented a method which is considered as a sequential form of the Schwarz Dirichlet-Neumann method. Elleithy and Al-Gahtani [24] presented an overlapping iterative domain decomposition method for

coupling of the FEM and BEM. The domain of the original problem is subdivided into a FEM sub-domain, a BEM sub-domain, and a common region, which is modeled by both methods.

The above iterative coupling methods, however, are only limited to linear problems. The objective of this paper is to extend the application of the sequential Schwarz Dirichlet-Neumann iterative coupling method to elasto-plasticity. Applications in fracture mechanics are considered. The conventional FEM computations are also performed, and a critical comparison of the results is made.

2 Iterative Coupling Method in Elasto-Plasticity

In this section we consider the extension of the sequential Schwarz Dirichlet-Neumann iterative coupling method presented by Lin et al. [22], and Feng and Owen [23] to elasto-plasticity. As any other coupling procedure, the starting point is to decompose the domain of the original problem into two sub-domains B and F . Now, let us define the following vectors (Figure 1):

- $\{u_B\}$: displacement in the BEM sub-domain,
- $\{u_B^I\}$: displacement on the BEM/FEM interface (but it is approached from the BEM sub-domain),
- $\{u_B^B\}$: displacement in the BEM sub-domain except $\{u_B^I\}$,
- $\{u_B\} = \{u_B^B, u_B^I\}$
- $\{u_F\}$: displacement in the FEM sub-domain,
- $\{u_F^I\}$: displacement on the BEM/FEM interface (approached from the FEM sub-domain), and

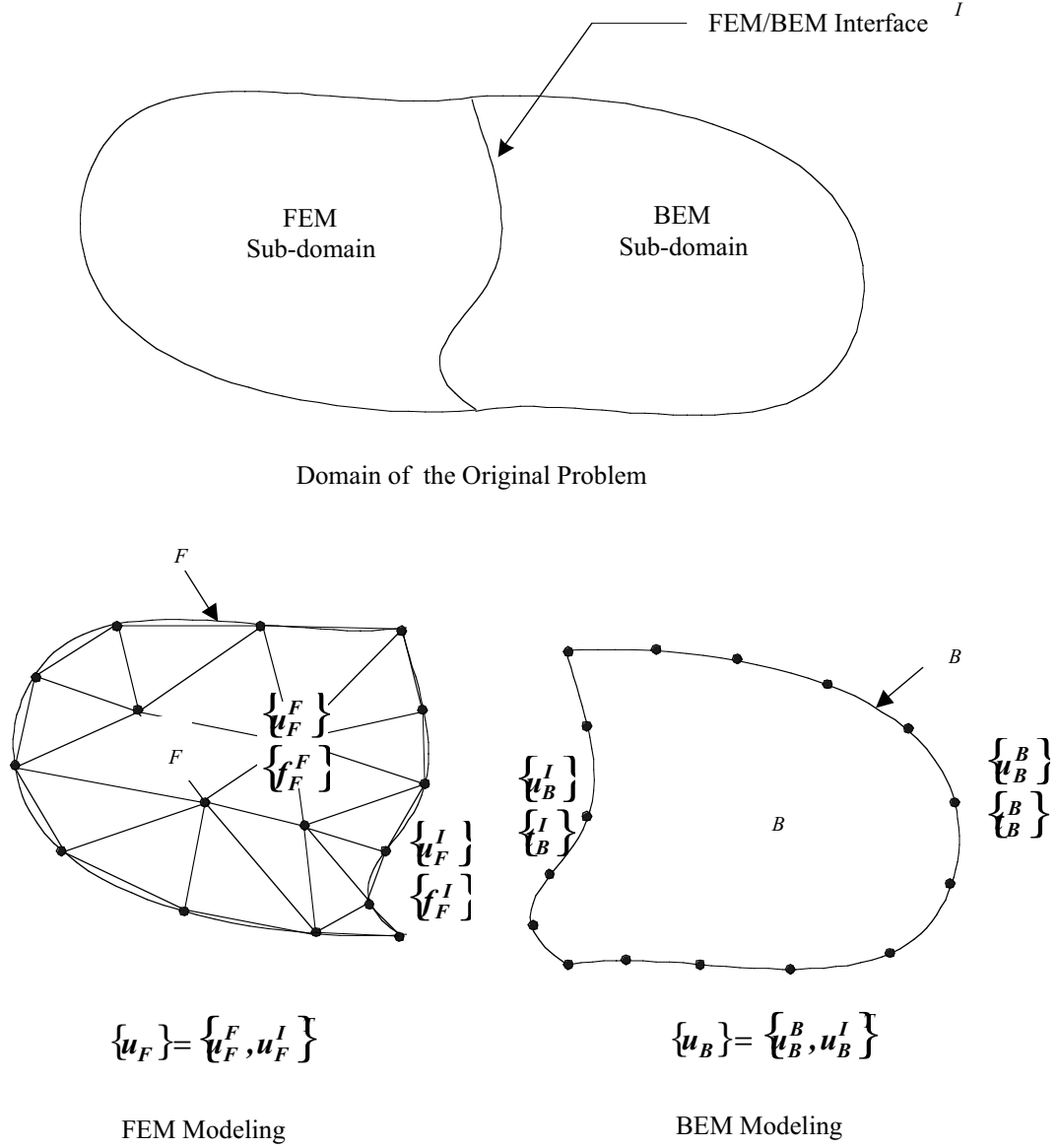


Figure 1: Domain Decomposition

$\{u_F^F\}$: displacement in the FEM sub-domain except $\{u_F^I\}$,
 $\{u_F\} = \{u_F^F, u_F^I\}$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_F^F \\ u_F^I \end{bmatrix} = \begin{bmatrix} f_F^F \\ f_F^I \end{bmatrix} \quad (2)$$

Similarly, one can denote the BEM traction by t_B^B and t_B^I and FEM force vectors by f_F^F and f_F^I . Disregarding body forces, the assembled boundary element equations for an elastic region are given by:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_B^B \\ u_B^I \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} t_B^B \\ t_B^I \end{bmatrix} \quad (1)$$

For an elasto-plastic analysis, the incremental form of the FEM equations can be written as:

It should be noted that for each load increment, Equations (2) are nonlinear and therefore are solved iteratively. At the interface, the compatibility and equilibrium conditions should be satisfied, i.e.,

$$\{u_B^I\} = \{u_F^I\} \quad (3)$$

$$\{f_F^I\} + [M]\{t_B^I\} = 0 \quad (4)$$

where, $[M]$ is the converting matrix due to the weighing of the boundary tractions by the interpolation functions on the interface.

The iterative coupling method can be summarized as follows:

1. Given the initial guess $\{u_{B,0}^I\} = \{\bar{u}\}$.
2. For $n = 0, 1, 2, \dots$, do
 - Solve Equation (1) and get $\{u_{B,n}^I\}$
 - Solve Equation (4) and obtain $\{f_{F,n}^I\}$
 - For $i = 1, 2, \dots$, specified number of increments
 - Solve Equation (2) for $\{u_{F,i}^I\}$
 - Apply $\{u_{F,i+1}^I\} = \{u_{F,i}^I\} + \{\Delta u_{F,i}^I\}$
 - Obtain $\{u_{F,n}^I\}$
 - Apply $\{u_{B,n+1}^I\} = (1 - \alpha)\{u_{B,n}^I\} + \alpha\{u_{F,n}^I\}$

where α is a relaxation parameter
 - Until $\frac{\| \{u_{B,n+1}^I\} - \{u_{B,n}^I\} \|}{\| \{u_{B,n+1}^I\} \|} < \epsilon$ (given tolerance)

3. Applications

A coupled Fortran computer program has been developed for the iterative FEBEM elasto-plastic analysis using the ideas presented in Section 2. Two simple numerical examples in linear elastic fracture mechanics (LEFM) and elasto-plastic fracture mechanics (EPFM) are considered. The conventional FEM computations are also performed, and a critical comparison of the results is made.

It should be noted that for the LEFM example in Section 3.1, the BEM is expected to give more accurate results than the FEM, as the BEM accurately capture the singular behavior at the crack tip. However, we conducted the analysis using the FEBEM and the FEM for the LEFM problem to account for cases where the rest of the domain may be non-homogenous or non-linearity is present. The reason behind choosing this simple LEFM example is to compare the results with the available exact solution.

3.1. Linear Fracture Mechanics Example

Consider a square plate, with a central crack, subjected to a uniform applied traction on the opposite ends of the plate (Figure 2-a). This produces a Mode I type of crack growth. The crack is assumed to be 10 units long with a plate width of 20 units. Young's modulus is assumed to be 0.3×10^5 units and a Poisson's ratio $\nu = 0.3$. A uniform traction of 1 unit is applied at opposite ends of the plate. Due to the symmetrical nature of the problem, only a quarter of the plate is modeled. The discretization of the linear FEBEM model is shown in Figure 2-b. The problem is modeled with 42 non-uniform linear boundary elements and 30 finite linear quadrilateral elements. The same problem is solved using the FEM with 682 linear quadrilateral elements. The stress intensity factors using the FEBEM and FEM are shown in Table 1. Notice that the FEBEM gives a stress

intensity factor that is only 1.9% different than the analytical solution, while the FEM gives an error of 5.9%. The difference in CPU time recorded for both methods is insignificant and therefore a comparison of the results is not given here.

3.2. Non-Linear Fracture Mechanics Example

Coupling the FEM and BEM may be most efficient for EPFM problems as the material is plastic around the crack and the FEM is more efficient in modeling the nonlinear regions. The remaining linear elastic region can be modeled by the BEM.

The geometry and loading assumed in this example is shown in Figure 3. Von Mises yield criterion is assumed and the material properties employed are as follows: Young's modulus $E = 2.06 \times 10^5$ units, Poisson's ratio $\nu = 0.3$, tensile yield stress $\sigma_y = 480$ units, and the tangent modulus for plasticity $H = 2.06 \times 10^3$ units. Due to symmetry only one quarter of the plate is modeled. The FEM and FEBEM analysis are performed with the discretization shown in Figure 4. Table 2 shows the computed remote stress σ_o vs.

load-point displacement δ_o . The calculated yield zones are also shown in Figure 5. The data in Table 2 and Figure 5 exhibit close agreement of results between the FEM and FEBEM. Table 3 shows the CPU time required for the analysis with the FEM and FEBEM. The Table shows a less CPU time when the analysis is performed using the FEM. The difference in CPU time increases as load increases. Utilizing a parallel processing for the iterative method is expected to result in a reduction of the CPU time required for analysis using FEBEM and it will be considered for future research. However, an advantage of the FEBEM which, cannot be seen from the results is the incredible reduction of data preparation required for analysis as compared to the FEM

4. Conclusions

The extension of the iterative coupling of FEM and BEM to elasto-plasticity is investigated in this paper. Beside the convenience of less input data, the iterative FEBEM has the advantage of preserving the identity of both FEM and BEM and therefore different formulation can be adopted for each method without changing the overall structure of the computer codes. The numerical examples show that the iterative FEBEM, in general, yields more accurate results as compared to the FEM.

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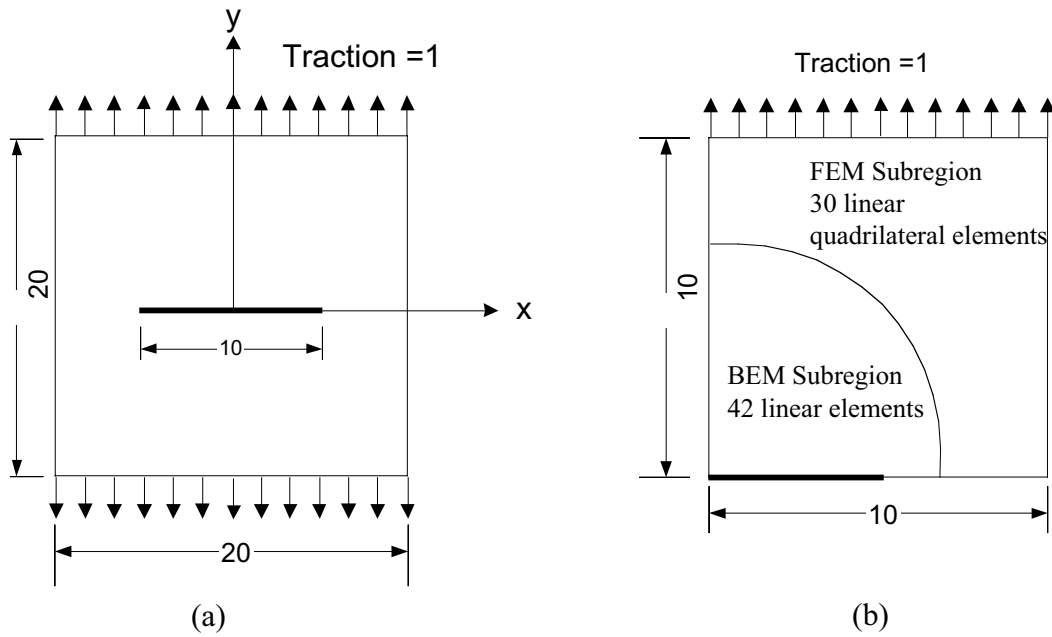
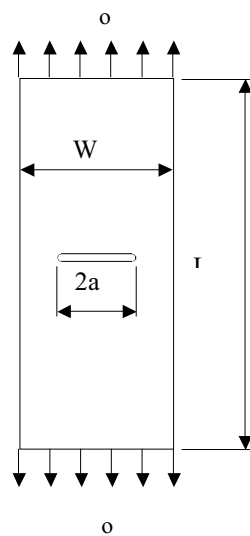


Figure 2: (a) Plate with a Central Crack (b) FEBEM Discretization.

Table 1: Stress Intensity Factors for a Cracked Plate.

Method	Stress Intensity Factor	% Error
Exact	4.71	-
FEM	4.28	5.9
FEBEM	4.62	1.9

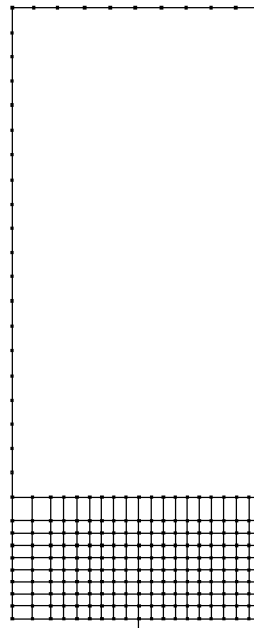
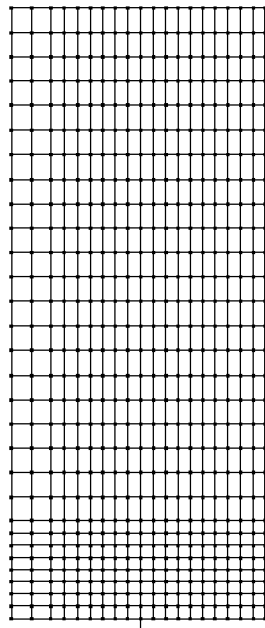


Von Mises yield Criterion
 Plane Strain
 $E = 2.06 \times 10^5$ units
 $\nu = 0.3$
 $\sigma_y = 480$ units
 $H' = 2.06 \times 10^3$ units
 $W = 40$ units
 $L = 100$ units
 $2a = 20$ units

Figure 3: Geometry and Loading Condition for Elasto-Plastic Fracture Mechanics Example

FEM
551 Finite Elements

FEBEM
171 Finite Elements
69 Boundary Elements



Crack Tip

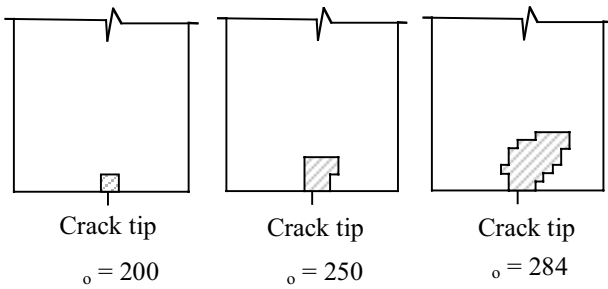
Crack Tip

Figure 4: Discretization for Elasto-Plastic Fracture Mechanics Example

Table 2: Remote Stress vs. Load-Point-Displacement for Elasto-Plastic Fracture Mechanics Example

σ_0 (units)	σ_0/W	
	FEM	FEBEM
100	0.065×10^{-2}	0.065×10^{-2}
150	0.098×10^{-2}	0.096×10^{-2}
200	0.131×10^{-2}	0.130×10^{-2}
226	0.149×10^{-2}	0.146×10^{-2}
250	0.166×10^{-2}	0.162×10^{-2}
284	0.194×10^{-2}	0.192×10^{-2}

FEM



FEBEM

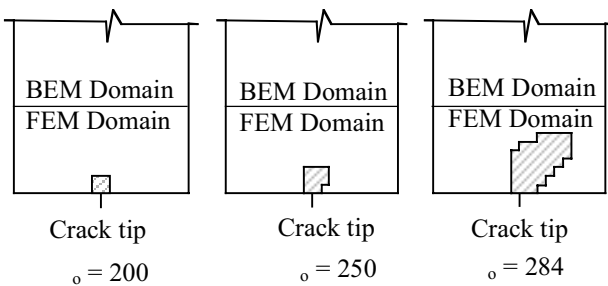


Table 3: CPU Time for Elasto Plastic Fracture Mechanics Example

σ_0	CPU time (Sec.)	
	FEM	FEM/BEM
100	3	5
150	4	5
200	4	8
226	5	8
250	5	8
284	6	11

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Figure 5: Yielded Zones for Elasto-Plastic Fracture Mechanics Example

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