

# Iterative coupling of BE and FE methods in elastostatics

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## Abstract

Iterative domain decomposition coupling is one of the recent approaches for combining the boundary element method (BEM) and the finite element method (FEM). The domain of the original problem is subdivided into two sub-domains, which are separately modeled by the FEM and BEM. Successive renewal of the variables on the interface of the two sub-domains is performed through an iterative procedure to reach the final convergence. In this paper, we investigate the iterative method. We also establish the convergence conditions. A simple numerical example is given to elaborate on the effect of different factors such as initial guess, boundary conditions, and geometrical and material properties of the sub-domains on solution convergence. © 2001 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

The boundary element method (BEM) and the finite element method (FEM) are widely used for the numerical solution of partial differential equations. Each of the two methods has its own advantages and disadvantages, and there are undoubtedly situations, which favor FEM over BEM and vice versa. Often one problem can give rise to a model favoring one method in one region and the other method in another region. Examples include the detailed analysis of stresses around an underground opening. The FEM can be employed to capture the plastic behavior at the vicinity of the opening. The remaining infinite/semi-infinite linear elastic region may be best represented by the BEM. Another application, where coupling the FEM and BEM seems to be most efficient, is the analysis of elasto-plastic fracture mechanics problems. In such cases, the plastic region around the crack can be modeled by the FEM while the remaining linear elastic region can be modeled by the BEM.

Unfortunately, the system of equations, produced by the FEM and BEM, are expressed in terms of different variables and cannot be linked as they stand. The coupling of the two methods has been a topic of great interest for more than two decades. It was achieved using various approaches, which

can be roughly classified into FEM hosted, BEM hosted, and those not belonging to either of these two categories. See, e.g. Refs. [1–27] not to mention many others. A literature survey on the subject can be found elsewhere [14,28,29].

The first two approaches [1–15] employ an entire unified equation for the whole domain, by combining the discretized equations for the BEM and FEM sub-domains. Although the FEM hosted approach conceived more convenience than the BEM hosted approach, their shortcoming is that the algorithm for constructing the entire equation is highly complicated when compared with that for each single equation. In order to overcome the stated inconvenience, iterative domain decomposition coupling methods were developed [21–27] where there is no need to combine the coefficient matrix for the FEM and BEM sub-domains.

Although, the iterative coupling methods offer many advantages over other methods and seems to be promising, the important issue of convergence is still not fully addressed. The objective of this paper is to establish the convergence conditions of the iterative method proposed by Lin et al. [25] and Feng and Owen [26]. We also discuss the effect of various factors involved in the convergence of the iterative coupling method.

## 2. BEM and FEM algebraic representation

Consider an isotropic, linear elastic solid of domain  $\Omega$

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enclosed by a boundary  $\Gamma$ . Somigliana's identity is the starting point for stress analysis of elastic bodies by direct BEM. This identity can be written as [30]:

$$C_{ij}(\zeta)u_j(\zeta) = \int_{\Gamma} u_{ij}^*(\zeta, x)t_j(x) d\Gamma - \int_{\Gamma} t_{ij}^*(\zeta, x)u_j d\Gamma + \int_{\Omega} u_{ij}^*(\zeta, x)b_j(x) d\Omega \quad (1)$$

where the indicial notation and Cartesian reference frame are used. The quantities  $u_j$ ,  $t_j$  and  $b_j$  denote components of the displacement, traction and body force vectors, respectively. The symbol '\*' stands for the fundamental values and the free term  $C_{ij}$  is dependent upon the geometry of the boundary.

For the numerical implementation of Eq. (1), the boundary  $\Gamma$  is divided into a number of boundary elements  $\Gamma^e$  ( $\Gamma = \sum_e \Gamma^e$ ) and the geometry of each element and the displacements and tractions on it are approximated by different interpolation functions. Neglecting body forces and after the process of boundary discretization, the boundary elements equations are obtained

$$[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{t}\} \in \Gamma^B \quad (2)$$

where  $\{\mathbf{u}\}$  and  $\{\mathbf{t}\}$  are vectors containing the boundary values for the displacements and the tractions and  $[\mathbf{H}]$  and  $[\mathbf{G}]$  are influence coefficient matrices.

For the FEM, the algebraic assembled element equations are given by

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{f}\} \in \Omega^F \quad (3)$$

where  $[\mathbf{K}]$  is the stiffness matrix for the system, and  $\{\mathbf{u}\}$  and  $\{\mathbf{f}\}$  are the nodal displacements and force vectors, respectively.

### 3. Iterative coupling methods

As mentioned earlier, the conventional coupling methods employ an entire unified equation for the whole domain by altering the formulation of one of the methods to make it compatible with the other. However, the algorithm for constructing the entire equation is highly complicated when compared with that of each single equation. Another drawback of the conventional coupling methods is that the resulting assembled matrix is asymmetric and fully populated in contrast to the symmetric sparsely banded stiffness matrix of the FEM. Hence, special matrix equation solvers have to be used, which are less efficient than the symmetric equation solvers.

More recently, the coupling of the BEM and the FEM has been achieved through the iterative domain decomposition methods [21–27]. In these coupling methods, there is no need to combine the coefficient matrices for the FEM and the BEM sub-domains, as required in most of the conventional coupling methods. A second advantage is that different formulation for the FEM and BEM can be adopted as

base programs for coupling the computer codes only. Separate computing for each sub-domain and successive renewal of the variables on the interface of both sub-domains are performed to reach the final convergence.

Gerstle et al. [21] presented a solution method, which is iterative in nature. Their idea is to iteratively apply displacement boundary conditions on the interface of the FEM and BEM sub-domains, calculate the resulting forces on the interface, and then to use the unbalanced force vector at the interface as a predictor for the applied displacements in the next iteration. However, their method is applicable only for symmetric BEM formulation. Perera et al. [22] presented a parallel method, which is based on the interface equilibrium of Steklov Poincare. Their method may not be suited for certain classes of problems where the natural boundary conditions are specified for the entire external boundary of the FEM or BEM sub-domains. In such case the specification of Neumann boundary conditions over the whole boundary of any sub-domain, will result in non-unique solutions. Kamiya et al. [23] employed the renewal methods known as Schwarz Neumann–Neumann and Schwarz Dirichlet–Neumann. Both methods, however, are not suited for problems where the natural boundary conditions are specified on the entire external boundary of the FEM sub-domain. Kamiya and Iwase [24] introduced an iterative analysis using conjugate gradient and condensation. However, the conjugate gradient method is only applicable to symmetric BEM formulation. Lin et al. [25], and Feng and Owen [26] proposed a method which is considered as a sequential form of the Schwarz Dirichlet–Neumann method. The method is based on assigning an arbitrary displacement vector to the interface for the BEM sub-domain. Then, the energy equivalent nodal forces of the solved interface tractions are treated as boundary conditions for the FEM sub-domain to solve for the interfacial displacements. The solution is achieved when these two sets of displacements converge. Elleithy and Al-Gahtani [27] presented an overlapping iterative domain decomposition method for coupling of the FEM and BEM. The domain of the original problem is subdivided into a FEM sub-domain, a BEM sub-domain, and a common region, which is modeled by both methods.

To the best of the authors' knowledge, the existing iterative domain decomposition methods for coupling the FEM and BEM, did not consider the case of multi-domain coupling or those with very complicated geometry. However, it is possible to extend the existing iterative methods to account for such cases. Also, it is not out of place to mention that, all of the existing iterative coupling methods require the definition of certain parameters, which may be empirically selected. Some trial and error and extensive experience are inevitable. Here comes the contribution of this paper where we establish the convergence conditions of the iterative method presented by Lin et al. [25] and Feng and Owen [26]. We also provide a simple guide for the choice of such parameter.

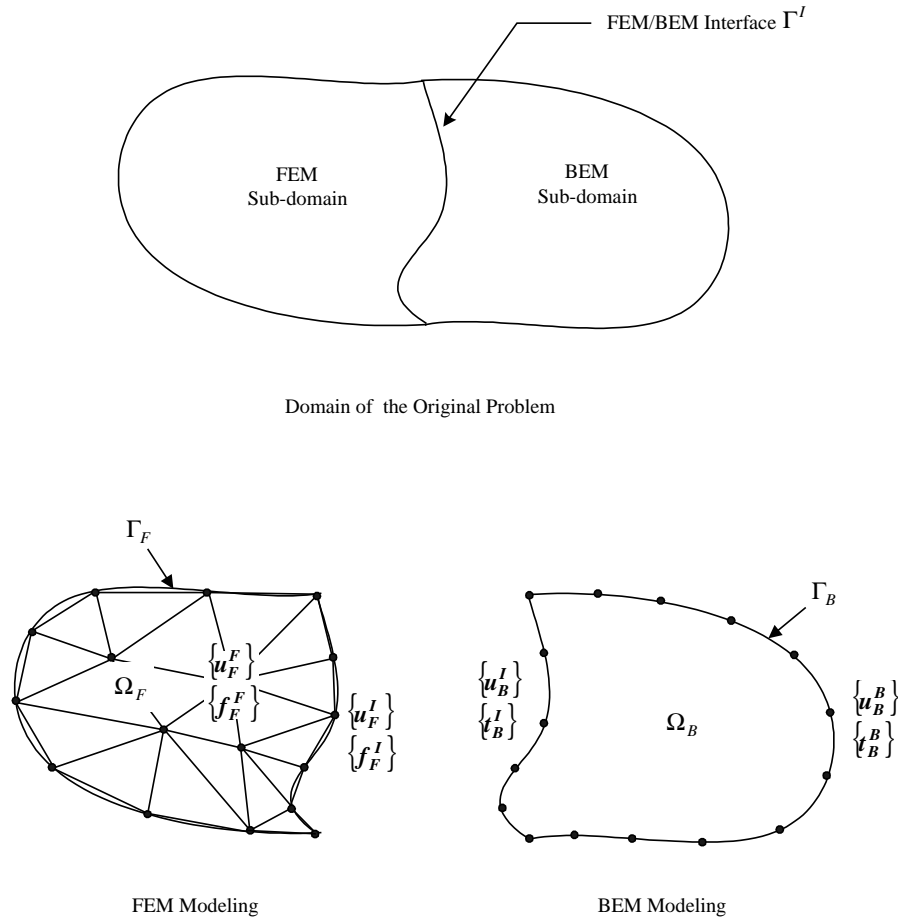


Fig. 1. Domain decomposition.

In the remainder of this section, we review the formulation of the iterative domain decomposition coupling method presented by Lin et al. [25] and Feng and Owen [26].

Consider Fig. 1, where the domain of the original problem is decomposed into two sub-domains  $\Omega^B$  and  $\Omega^F$ . Now, let us define the following vectors (Fig. 1):

$\{\mathbf{u}_B\}$  is the displacement in the BEM sub-domain,  $\{\mathbf{u}_B^I\}$  the displacement on the BEM/FEM interface (but it is approached from the BEM sub-domain),  $\{\mathbf{u}_B^B\}$  the displacement in the BEM sub-domain except  $\{\mathbf{u}_B^I\}$ ,

$$\{\mathbf{u}_B\} = \{\mathbf{u}_B^B, \mathbf{u}_B^I\}^T$$

$\{\mathbf{u}_F\}$  the displacement in the FEM sub-domain,  $\{\mathbf{u}_F^I\}$  the displacement on the BEM/FEM interface (approached from the FEM sub-domain), and  $\{\mathbf{u}_F^F\}$  is the displacement in the FEM sub-domain except  $\{\mathbf{u}_F^I\}$ ,

$$\{\mathbf{u}_F\} = \{\mathbf{u}_F^F, \mathbf{u}_F^I\}^T$$

Similarly, one can define the traction and force vectors for the BEM and the FEM sub-domains, respectively.

Eqs. (2) and (3) may be partitioned as follows:

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^B \\ \mathbf{u}_B^I \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_B^B \\ \mathbf{t}_B^I \end{Bmatrix} \quad (4)$$

and

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_F^F \\ \mathbf{u}_F^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_F^F \\ \mathbf{f}_F^I \end{Bmatrix} \quad (5)$$

At the interface, the compatibility and equilibrium conditions are

$$\{\mathbf{u}_B^I\} = \{\mathbf{u}_F^I\} \in \Gamma^I \quad (6)$$

$$\{\mathbf{f}_F^I\} + [\mathbf{M}]\{\mathbf{t}_B^I\} = 0 \in \Gamma^I \quad (7)$$

where  $[\mathbf{M}]$  is the converting matrix due to the weighing of the boundary tractions by the interpolation function on the interface.

The iterative coupling method [25,26] can be summarized as follows:

1. The problem domain is subdivided into two sub-domains that are well behaved and solvable and are modeled by the FEM and BEM methods.

2. Set initial values of  $\{\mathbf{u}^I\}$  at the interface, i.e.  $\{\mathbf{u}_B^I\} = 0$ .
3. Consider the BEM sub-domain and solve Eq. (4) for the traction  $\{\mathbf{t}_B^I\}$  at the interface.
4. The corresponding nodal forces on the interface for the FEM sub-domain can be determined using Eq. (7).
5. Solve for the displacements at the interface for the finite element sub-domain,  $\{\mathbf{u}_F^I\}$ , using Eq. (5)
6. Check for convergence at the interface, i.e.

$$\frac{\|\{\mathbf{u}_{B,n+1}^I\} - \{\mathbf{u}_{B,n}^I\}\|}{\|\{\mathbf{u}_{B,n+1}^I\}\|} < \epsilon$$

- where  $\epsilon$  is a predefined tolerance. If convergence is achieved then stop, if not, for the next iteration set  $\{\mathbf{u}_{B,n+1}^I\} = (1 - \alpha)\{\mathbf{u}_{B,n}^I\} + \alpha\{\mathbf{u}_{F,n}^I\}$ , where  $\alpha$  is a relaxation parameter to speed up convergence.
7. Repeat steps 3 to 6 until convergence is achieved.

#### 4. Convergence of the iterative method

In this section, we wish to investigate the convergence of the iterative method depicted in Section 3. We shall establish that under certain conditions, and for sufficiently small  $\alpha$ , the iterations

$$\{\mathbf{u}_{B,n+1}^I\} = (1 - \alpha)\{\mathbf{u}_{B,n}^I\} + \alpha\{\mathbf{u}_{F,n}^I\} \tag{8}$$

will converge to the true value of  $\{\mathbf{u}^I\}$ . Our findings will be confirmed by the numerical example in Section 5.

After applying boundary conditions and rearranging, Eq. (4) can be written in the following form:

$$\begin{Bmatrix} \mathbf{X}_B^B \\ \mathbf{t}_B^I \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_B \\ \mathbf{u}_B^I \end{Bmatrix} \tag{9}$$

where  $\mathbf{X}_B^B$  are the boundary unknowns in the BEM sub-domain except at the interface. Similarly one can apply boundary conditions and rearrange Eq. (5) to obtain

$$\begin{Bmatrix} \mathbf{u}_F^I \\ \mathbf{f}_F^I \end{Bmatrix} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_F \\ \mathbf{u}_F^I \end{Bmatrix} \tag{10}$$

Note that  $\mathbf{C}_B$  and  $\mathbf{C}_F$  are vectors of constant values. Let us rewrite the second set of equations (9) and (10), respectively, as

$$\mathbf{t}_B^I = \mathbf{A}_{IB}\mathbf{C}_B + \mathbf{A}_{II}\mathbf{u}_B^I \tag{11}$$

and

$$\mathbf{u}_F^I = \mathbf{F}_{IF}\mathbf{C}_F + \mathbf{F}_{II}\mathbf{f}_F^I \tag{12}$$

Eliminating  $\mathbf{f}_F^I$  from Eq. (12) using Eq. (7), yields

$$\mathbf{u}_F^I = \mathbf{F}_{IF}\mathbf{C}_F - \mathbf{F}_{II}\mathbf{M}\mathbf{t}_B^I \tag{13}$$

Similarly, eliminating  $\mathbf{t}_B^I$  from Eq. (13) using Eq. (11),

yields

$$\mathbf{u}_F^I = \mathbf{C}\mathbf{u}_B^I + \mathbf{c} \tag{14}$$

where

$$\mathbf{C} = -\mathbf{F}_{II}\mathbf{M}\mathbf{A}_{II}$$

and

$$\mathbf{c} = \mathbf{F}_{IF}\mathbf{C}_F - \mathbf{F}_{II}\mathbf{M}\mathbf{A}_{IB}\mathbf{C}_B$$

Substituting for  $\mathbf{u}_{F,n}^I$  in Eq. (8), using Eq. (14), we get

$$\mathbf{u}_{B,n+1}^I = [(1 - \alpha)\mathbf{I} + \alpha\mathbf{C}]\mathbf{u}_{B,n}^I + \alpha\mathbf{c} \tag{15}$$

Now, Eq. (15) represents an iterative method of the form:

$$\mathbf{X}_{n+1} = \mathbf{D}_\alpha\mathbf{X}_n + \mathbf{d}_\alpha \tag{16}$$

which converges if and only if the spectrum  $\sigma(\mathbf{D}_\alpha)$  of the matrix  $\mathbf{D}_\alpha$  is contained in the unit sphere  $B(0, 1)$  in the complex plane [31], i.e. Eq. (15) converges if and only if

$$\sigma((1 - \alpha)\mathbf{I} + \alpha\mathbf{C}) \subseteq B(0, 1)$$

or

$$\sigma(\alpha\mathbf{C}) \subseteq B(\alpha - 1, 1)$$

or

$$\sigma(\mathbf{C}) \subseteq B\left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right)$$

The sphere is centered at  $Z = 1 - (1/\alpha)$  with the radius  $1/\alpha$ .

Next we will show that if  $\lambda = x + iy \in \sigma(\mathbf{C})$  with  $x < 1$ , then  $\lambda \in B(1 - (1/\alpha), 1/\alpha)$  for some  $\alpha$ . For this we have to satisfy the inequality:

$$\left| \lambda - \left(1 - \frac{1}{\alpha}\right) \right| < \frac{1}{\alpha}$$

or

$$\left[ x - \left(1 - \frac{1}{\alpha}\right) \right]^2 + y^2 < \frac{1}{\alpha^2}$$

which upon simplification, gives

$$(1 - x)^2 + y^2 < \frac{2(1 - x)}{\alpha} \tag{17}$$

Note that inequality (17) immediately implies the necessary condition  $x < 1$ . Inequality (17) can be rewritten as

$$\alpha < \frac{2(1 - x)}{(1 - x)^2 + y^2}$$

The foregoing discussion shows that if  $\lambda_1 = x_1 + iy_1, \dots, \lambda_N = x_N + iy_N$  are the eigenvalues of  $\mathbf{C}$ , then

$$\alpha < \min_{1 \leq i \leq N} \left\{ \frac{2(1 - x_i)}{(1 - x_i)^2 + y_i^2} \right\} \quad \text{and} \tag{18}$$

$$x_i < 1, \quad i = 1, 2, \dots, N$$

are the necessary conditions for the convergence of the iterative procedure.

Next we will show that for a proper choice of  $\alpha$  one may minimize the spectral radius of the iteration matrix  $((1 - \alpha)\mathbf{I} + \alpha\mathbf{C})$ , which we denote by  $\rho((1 - \alpha)\mathbf{I} + \alpha\mathbf{C})$ .

Let  $\boldsymbol{\lambda}^t = (\lambda_1 \lambda_2 \dots \lambda_n)$  and  $\mathbf{1}^t = (1 \ 1 \dots 1)$ , then

$$\begin{aligned} \rho((1 - \alpha)\mathbf{I} + \alpha\mathbf{C}) &= \max_{1 \leq i \leq N} \{ |(1 - \alpha) + \alpha\lambda_i| \} \\ &= \|(1 - \alpha)\mathbf{1} + \alpha\boldsymbol{\lambda}\|_\infty \end{aligned}$$

The problem now is to choose  $\alpha$  such that  $\|(1 - \alpha)\mathbf{1} + \alpha\boldsymbol{\lambda}\|_\infty$  is minimized. Due to the fact that  $\|\cdot\|_\infty$  is not differentiable, an explicit value for the optimum  $\alpha$  is not readily obtainable. However, noting that  $(1/\sqrt{N})\|x\|_2 \leq \|x\|_\infty \leq \|x\|_2$ , one can try to obtain a value of  $\bar{\alpha}$  that minimizes the Euclidean norm. If for such an  $\bar{\alpha}$ , it turns out that  $\|(1 - \bar{\alpha})\mathbf{1} + \bar{\alpha}\boldsymbol{\lambda}\|_2 < 1$ , then so will be the infinity norm and consequently the spectral radius of our iteration matrix. Proceeding with this idea let  $F(\alpha) = \|(1 - \alpha)\mathbf{1} + \alpha\boldsymbol{\lambda}\|_2^2$ , then

$$F'(\alpha) = 2 \operatorname{Re}(\mathbf{1}^t(\boldsymbol{\lambda} - \mathbf{1})) + 2\alpha\|\boldsymbol{\lambda} - \mathbf{1}\|^2 \tag{19}$$

and

$F''(\alpha) = 2\|\boldsymbol{\lambda} - \mathbf{1}\|^2 > 0$ , i.e. the initial values (19) correspond to minimum  $\alpha$ . Now setting  $F'(\alpha) = 0$  we obtain

$$\bar{\alpha} = -\frac{\operatorname{Re}(\mathbf{1}^t(\boldsymbol{\lambda} - \mathbf{1}))}{\|\boldsymbol{\lambda} - \mathbf{1}\|^2} \tag{20}$$

and

$$F_{\min} = N - \frac{|\operatorname{Re}(\mathbf{1}^t(\boldsymbol{\lambda} - \mathbf{1}))|^2}{\|\boldsymbol{\lambda} - \mathbf{1}\|^2} \tag{21}$$

A sufficient condition for convergence then is  $F_{\min} < 1$ , for in this case. Moreover, this condition implies that  $\bar{\alpha}$  necessarily satisfies Eq. (18).

The two conditions for convergence depicted by Eq. (18) give rise to a set of factors that control convergence. The most important one is the selection of the parameter  $\alpha$ , which greatly affects convergence of the iterative method. Beyond the values given by Eq. (18), the iterative method will not converge. Also from the discussion given in this section, one can conclude that convergence is dependent on the eigenvalues of the matrix  $\mathbf{C}$ , which in turn are dependent on  $\mathbf{K}$ ,  $\mathbf{H}$ ,  $\mathbf{G}$  and  $\mathbf{M}$  matrices. This indicates that convergence is dependent on the mesh density of the problem sub-domains, specified boundary conditions, and the geometrical and material properties of the sub-domains. Although, the initial guess is not involved in the conditions for convergence it affects the speed of convergence of the scheme, as will be illustrated by the numerical example in Section 5.

### 5. Illustration

Conditions for convergence were established theoretically in Section 4. Also we demonstrated those factors that affect the convergence of the iterative method. Through a simple example, we wish to fix ideas presented in the previous section. It should be noted that the BEM/FEM coupling approach is versatile and capable of handling more complex problems than the example presented in this section, which only serves to clarify some issues related to the iterative method.

The steel cantilever beam shown in Fig. 2 is analyzed,

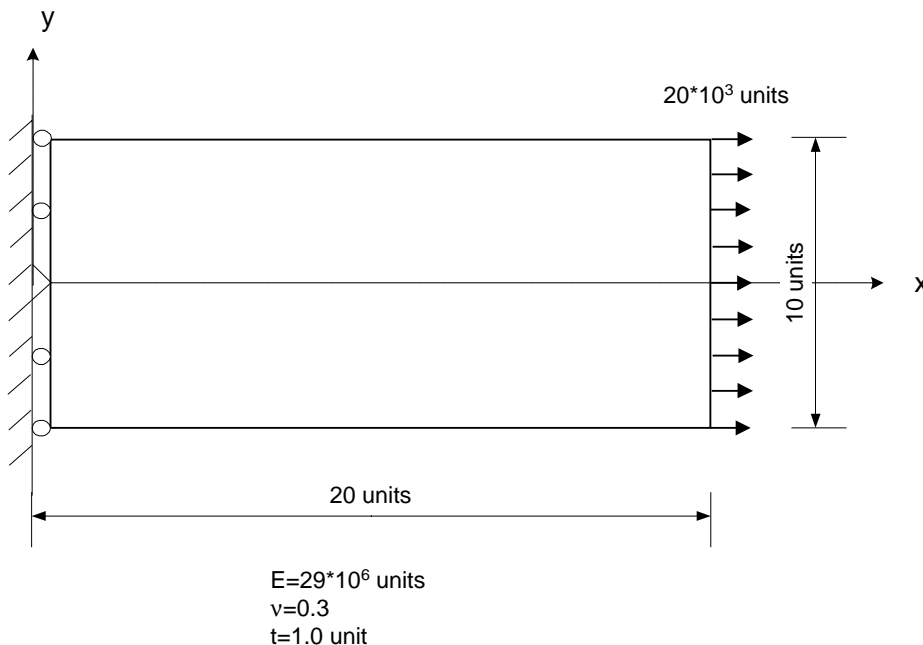


Fig. 2. Cantilever beam subjected to uniform loading.

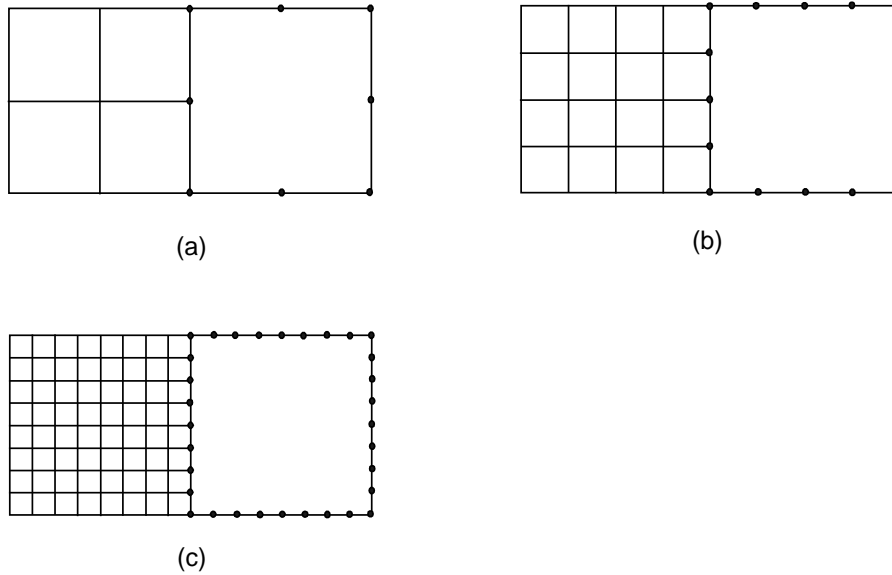


Fig. 3. BEM/FEM discretization.

and the results are compared with those from elasticity theory. The cantilever beam is subjected to a uniform tensile loading of  $20 \times 10^3$  units at its free end, and is considered to be in a state of plane stress with an elastic modulus,  $E = 29 \times 10^6$  units, and a Poisson's ratio  $\nu = 0.3$ . The beam is 20 units long and 10 units high, and is assumed to be weightless. The results obtained from the coupled BEM/FEM method using different meshes as shown in Fig. 3 match very well with the analytical solutions.

Fig. 4 illustrates the effect of the mesh size on the convergence of solution. For mesh (a),  $\alpha$  should be within the range of 0.03–0.72, whilst the range for mesh (c) is 0.03–0.47. Beyond these values the scheme will not converge.

The range from which the relaxation parameter to be chosen becomes narrower with a denser mesh of the computational sub-domains. Needless to say that, an optimal value of  $\alpha$  exists as this is a common knowledge. Two very important issues are to be observed over here. Although it is the same problem, different optimal values of  $\alpha$  exist for different FEM and BEM meshes. More importantly, one should be careful while choosing the value of  $\alpha$ , as some values will not assure the convergence of the scheme.

In order to address the effect of the geometry of the computational sub-domains on solution convergence, the problem is investigated for different relative areas of the finite and boundary element sub-domains, as shown in

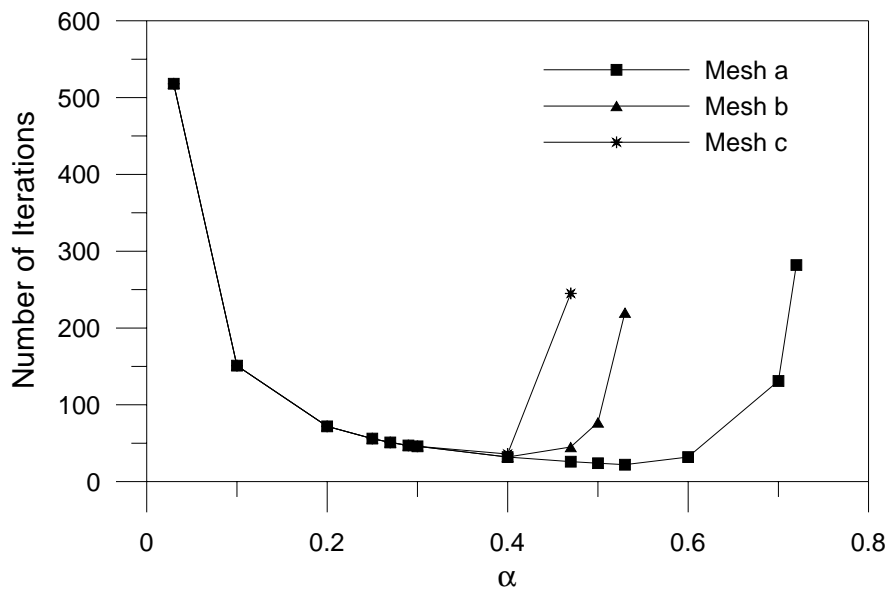


Fig. 4. Effect of the mesh density on solution convergence.

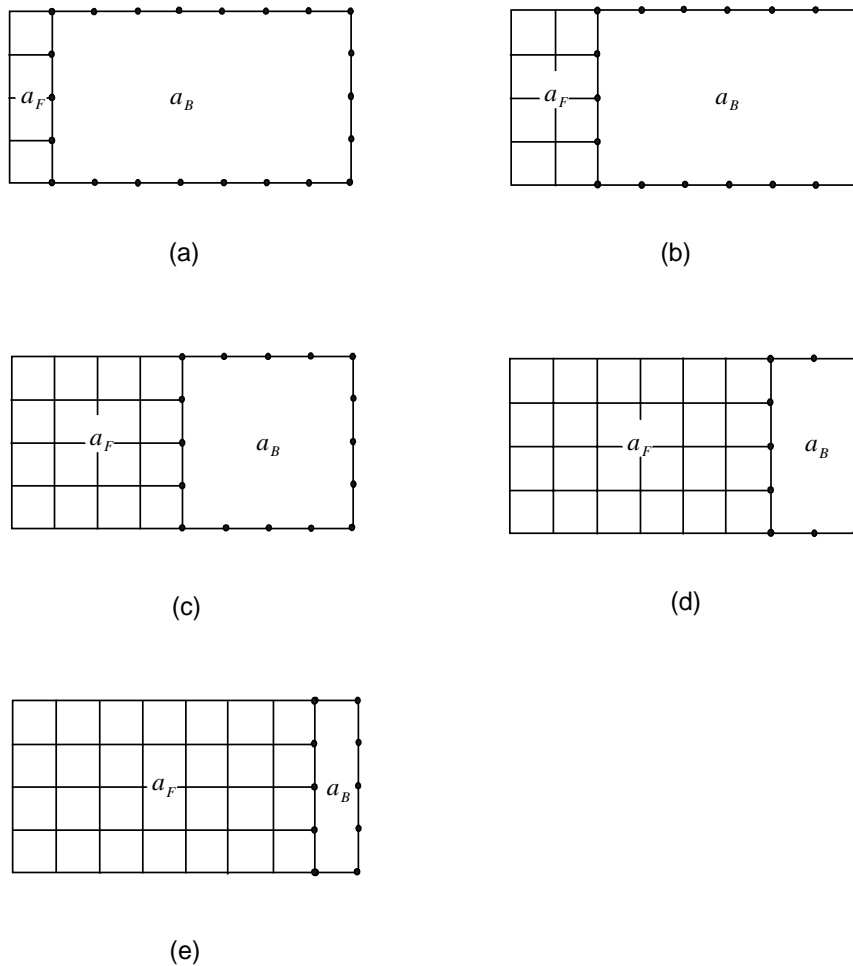


Fig. 5. Relative BEM to FEM computational sub-domains: (a)  $a_B/a_F = 7$ ; (b)  $a_B/a_F = 3$ ; (c)  $a_B/a_F = 1$ ; (d)  $a_B/a_F = 1/3$ ; (e)  $a_B/a_F = 1/7$ .

Fig. 5. From Fig. 6, it is observed that the minimum applicable range of  $\alpha$  is for problems having equal finite and boundary element areas,  $a_B/a_F = 1$ . As  $a_B/a_F$  increases or decreases, the applicable range of  $\alpha$  increases.

Using mesh (b) of Fig. 3, the problem is investigated for

different relative values of modulus of elasticity for the BEM and FEM sub-domains,  $E_B/E_F$ . Fig. 7 indicates that as  $E_B/E_F$  decreases, the range from which the parameter  $\alpha$  to be chosen increases. This range reduces to a very narrow one for higher values of  $E_B/E_F$ .

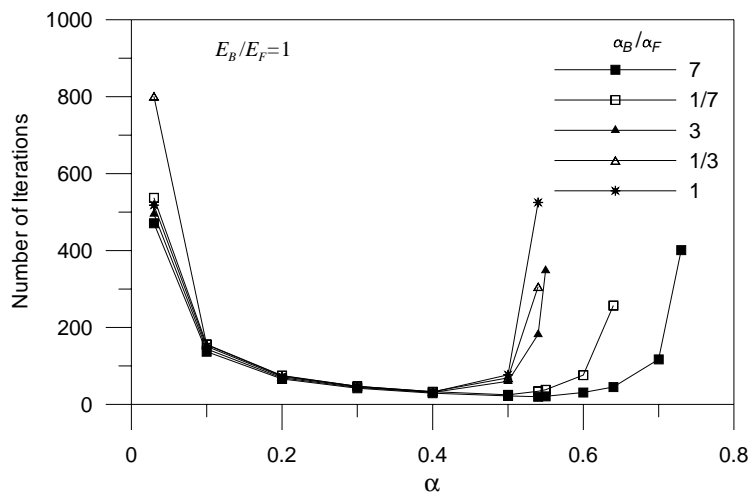


Fig. 6. Effect of the geometry of the computational sub-domains on  $\alpha$ .

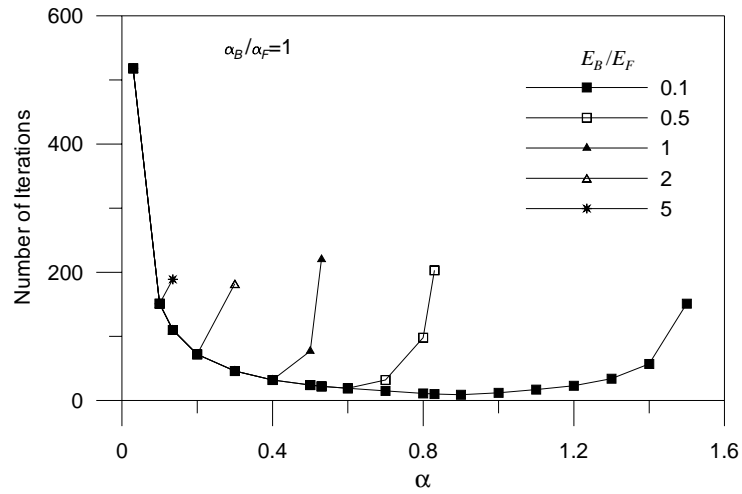


Fig. 7. Effect of material properties of the sub-domains on  $\alpha$ .

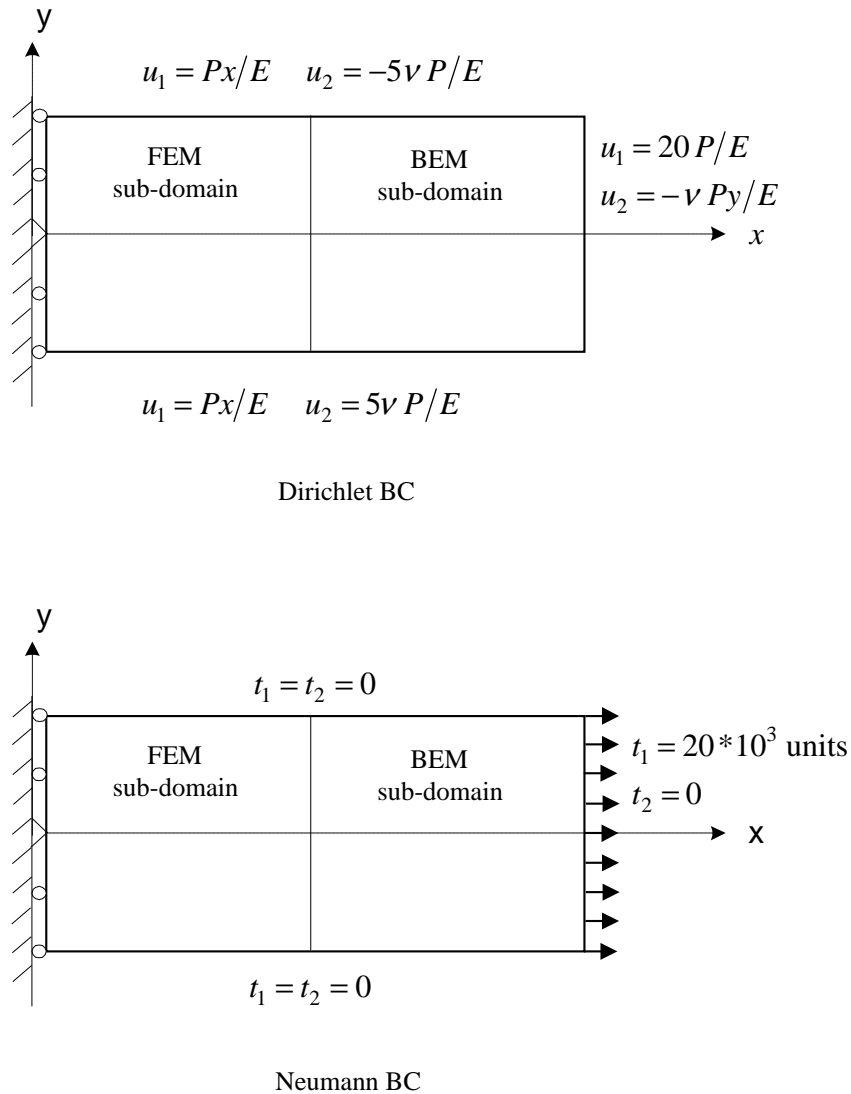


Fig. 8. Boundary conditions.



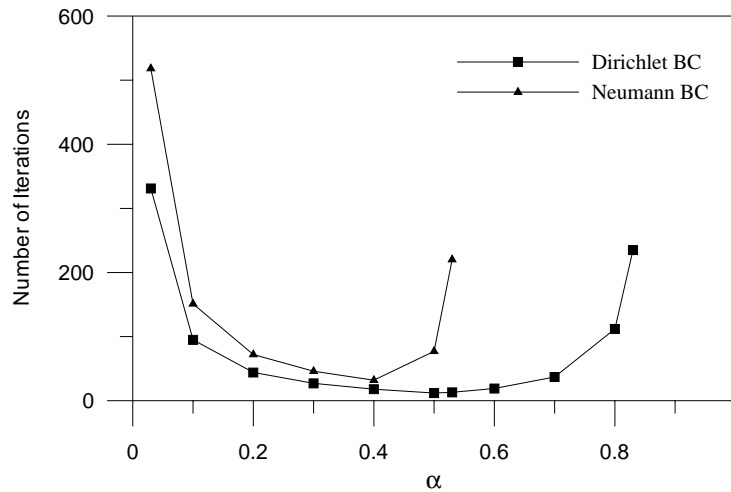


Fig. 9. Effect of boundary conditions on  $\alpha$ .

Using the two equivalent different types of boundary conditions as shown in Fig. 8, mesh (b) of Fig. 3 is also analyzed. As can be seen from Fig. 9, different types of boundary conditions results in different optimum value and applicable range of the parameter  $\alpha$ .

In order to investigate the effect of the initial guess of the unknowns at the interface, the same problem is reinvestigated using mesh (b) of Fig. 3 and different values of  $u^I$  as can be seen in Fig. 10. The results show that, the arbitrary assigned initial values of  $u^I$  only affect the speed of convergence. Obviously, a more reasonable guess yields less number of iterations. The allowable range will remain as 0.03–0.53 for this problem. It is reasonable to start with values of zeros at the interface for the initial displacements, which seems convenient as well as, appropriate from the physical realization.

It should be noted that the numerical results obtained in this section match with those of Eq. (18).

### 6. Selection of the relaxation parameter

The theoretical analysis and benchmark example presented in Sections 4 and 5, respectively, clearly identify the factors that control the convergence of the FEM/BEM iterative coupling method. These factors include the geometrical and material properties of the FEM and BEM sub-domains, specified type of boundary conditions of the sub-domains, and the mesh density of the FEM and BEM computational sub-domains. The most important issue regarding convergence is the selection of the parameter  $\alpha$ . Beyond the values given by Eq. (18) the iterative method does not converge.

It is also concluded that the initial guess is not involved in the conditions for convergence and it has an insignificant effect on the speed of convergence. A further examination of Eq. (18) indicates that the optimum value of  $\alpha$  is independent on the initial guess of the displacements on the FEM/

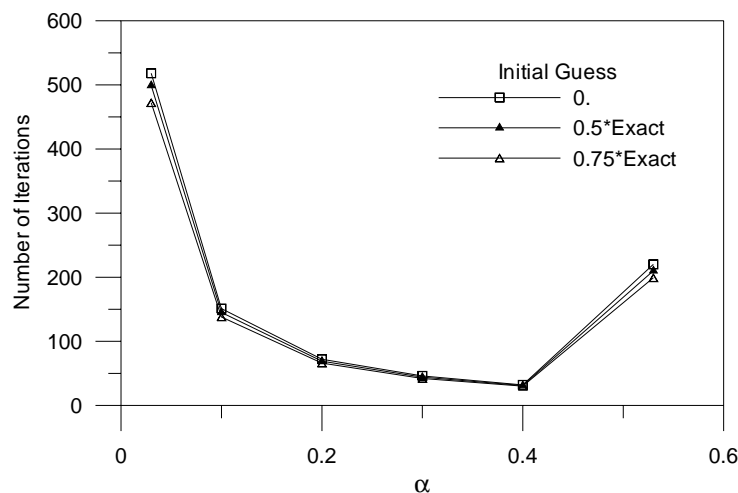


Fig. 10. Effect of the initial guess on  $\alpha$ .

BEM interface. This is also confirmed by the benchmark example. It is reasonable to start with values of zeros for the initial displacements on the interface.

For the selection of the parameter  $\alpha$ , Eqs. (18) and (20) should be utilized. However, it is very important to note that, by the complicated operators in Eq. (18) the behavior of the parameters for the solution convergence cannot be shown easily. In this paper, we established the convergence conditions. We also elaborated on the factors, which affect the solution convergence. The following guidelines may be helpful in selecting the parameter  $\alpha$ :

1. For combinations of low values of the relative sizes of the BEM to FEM sub-domains, and high values of the relative stiffness of the BEM to FEM sub-domains, the parameter  $\alpha$  is assigned a relatively low value.
2. For combinations of high values of the relative sizes of the BEM to FEM sub-domains, and low values of the relative stiffness of the BEM to FEM sub-domains, the applicable range of  $\alpha$  becomes wider.

Fortunately, most of the FEM/BEM coupling applications satisfy the second case and therefore,  $\alpha$  may be assigned a relatively higher value.

## 7. Conclusions

Unlike traditional methods of coupling, the concepts behind the iterative FEM/BEM coupling method have physical meaning and are easy to comprehend. In this paper, we established the general convergence conditions of the iterative method for elasticity problems. Results were verified through a simple numerical example. We also demonstrated those factors that affect solution convergence. Unlike the previous belief that the relaxation parameter is only needed to speed up the convergence, the theoretical analysis shows that this parameter controls the convergence of the iterative coupling method. Another interesting finding is that the derived convergence conditions do not involve the initial guess of the unknowns at the interface. This fact was also confirmed by the results of the numerical example. The initial guess had to do with the rate of convergence only.

The present study will be extended to consider problems involving complex geometry and material properties.

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