A meshless method for non-linear Poisson problems with high gradients

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A meshless method for the solution of linear and non-linear Poisson-type problems involving high gradients is presented. The proposed method is based on collocation with 3rd order polynomial radial basis function coupled with the fundamental solution. The linear problem is solved by satisfying the boundary conditions and the governing differential equations over selected points over the boundary and inside the domain, respectively. In the case of the non-linear case, the resulted equations are highly non-linear and therefore, they are solved using an incremental-iterative procedure. The accuracy and efficiency of the method is verified through several numerical examples.

Keywords: Meshless method, fundamental solution, RBF, high gradients

1. INTRODUCTION

The applications of meshless methods in computational mechanics have attracted much attention in recent years. These methods do not require mesh generation which is considered to be the most time-consuming part of a finite element formulation particularly in problems associated with frequent re-meshing. Although, the initial idea of mesh-less methods dates back to the smooth particle hydrodynamics (SPH) method by invented by Gingold in 1977 [1], the research into mesh-less methods has not become active until the development of the diffuse element method by Nayroles *et al.* [10]. Since then, several mesh-less methods have been reported in the literature [2, 7, 8, 11] to mention view. More details about these methods can be found in a book by Liu [6].

In terms of versatility and simplicity, the methods of collocations with radial basis functions (RBF) is the most appealing method. In addition to its simplicity, RBF does not involve the problem of imposition of essential boundary conditions as encountered in most other meshless methods. Furthermore, it is truly meshless as no elements are required for the interpolation of the solution variables. The roots of RBF goes back to the early 1970s, when it was used by Hardy [3] for fitting scattered data. In 1982, Nardini and Brebbia [9] applied RBFs to the dual reciprocity boundary element method to solve dynamic problems in solid mechanics, where the RBF was used to transform domain integral into boundary integral. Thereafter, many researchers have used RBF in conjunction with boundary element method (BEM) to solve various problems in computational mechanics. The method, however has not been applied directly to solve partial differential equations until 1990 by Kansa [4, 5]. Since then, many researchers have suggested several variations to the original method as reviewed in the book of Liu [7].

The simplest version of RBF is the direct collocation approach as proposed by Kansa [5] which is going to be referred to in this paper as DRBF. Although DRBF and other versions of RBF have been applied successfully different types of differential equations, their applicability to problems involving high gradients or singularities needs some attention. This paper tries to address that issue and suggests the addition of the logarithmic function as an augmented radial basis

function. The addition of such function serves two purposes. First, it behaves like a homogeneous solution for the major part of the differential equation, i.e. Laplace operator. Second, it has the capability of capturing the singularity of the solution, especially at some critical points on the boundary. The paper, also, suggests an incremental-iterative procedure for extending the proposed method to non-linear problems. Although, the formulation and application are given for 2-D Poisson-type differential equations, the procedure is applicable for other types of differential equations.

This paper is organized as follows. In Section 2, the formulation and numerical implementation of the direct collocation with radial basis functions is described. The method is extended to non-linear problems in Section 3. The application of the proposed method is discussed through several numerical examples in Section 4. The paper ends with conclusions in Section 5.

2. FORMULATION

Let us consider first the linear differential equation

$$L(\nabla^2 u, \nabla u, u) = f(x) \quad \text{in } \Omega \tag{1a}$$

where L is a second order linear differential operator and $f(x)$ is a continuous function of the position. Let us assume the following general boundary conditions,

$$B\left(u, \frac{\partial u}{\partial n}\right) = g(x) \quad \text{on } \partial\Omega, \tag{1b}$$

where B is a linear differential operator and $g(x)$ is a continuous function of the position. The solution of Eq. (1) can be approximated by

$$u(x) = \sum_{i}^{N} \alpha^i \Phi(\|x - x^i\|) + \sum_{j}^{M} \beta^j \Psi(\|x - x^j\|) + \sum_{k}^{L} \gamma^k \Theta(\|x - x^k\|) + \cdots \tag{2}$$

where $\|\cdot\|$ is the Euclidean distance, Φ, Ψ, and Θ are some suitable radial basis functions centered at x^i, x^j, x^k, respectively, and α, β, and γ are their respective coefficients to be determined. In the direct method of collocation with radial basis functions (DRBF), only the first term of the series is retained, i.e.

$$u(x) = \sum_{i}^{N} \alpha^i \Phi(\|x - x^i\|) \tag{3}$$

where Φ can be selected from but not limited to the following:

1. Nth Polynomial splines, R^n,

2. Multi-quadrics (MQ), $(c^2 + R^2)^{\frac{1}{2}}$,

3. Reciprocal multi-quadrics, $(c^2 + R^2)^{-\frac{1}{2}}$,

4. Gaussians, $\exp(-R^2/c)$,

where $R^2 = \|x - x^i\|$ and c is a shape parameter. Note that all except the polynomial require the specification of a shape factor, c. In order to limit the size of computations, only the first two RBFs are considered in this paper. The coefficients α^i can be obtained by using the approximated solution given by Eq. (3) to satisfy the differential equation (1a) at $N1$ nodes inside the domain and the

boundary condition (1b) at $N2$ boundary nodes, where $N1 + N2 = N$. The resulted collocation equations can be written in a matrix form as

L

$$
\begin{bmatrix} G(\Phi) \\ B(\Phi) \end{bmatrix} \begin{bmatrix} \alpha \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} . \tag{4}
$$

In the proposed modified RBF method (MRBF), we retain the first two terms in Eq. (2), i.e.

$$
u(x) = \sum_{i}^{N1+N2} \alpha^i \Phi(\|x - x^i\|) + \sum_{j}^{N1} \beta^j \Psi(\|x - x^j\|), \tag{5}
$$

where $\Psi = \mathrm{Log}\,\|x - x^j\|$ and the nodes x^j are distributed over a contour at an arbitrary distance from the boundary. It should be noted that Ψ is the fundamental solution corresponding to Laplace equation. The addition of this function is motivated by two factors. First, it satisfies Laplace operator and therefore serves as a partial homogeneous solution to the differential equation. Second, due to its singular behavior, it has the capability of capturing the singularity of the solution in case the problem involves high gradients on or close to the boundary. Although Φ can be chosen from the above mentioned RBFs, it is decided to use the cubic polynomial for two reasons: 1) It has no shape parameter and therefore, the procedure for optimizing the choice of that parameter is avoided, 2) Preliminary numerical experiments on several Poisson's equations have indicated that various radial basis functions gave the same accuracy when they have been combined with the logarithmic function.

The $2N1 + N2$ coefficients can be obtained by satisfying the differential equation (1a) at $N1$ selected points over the boundary plus $N2$ selected points inside the domain and satisfying the boundary conditions (1b) at the same $N1$ boundary nodes. The resulted system of equations can be written in the following matrix form

$$
\begin{bmatrix} L(\Phi) & 0 \\ B(\Phi) & B(\Psi) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} . \tag{6}
$$

In addition to the advantages mentioned above, the new method has another important advantage. It introduces a new $N1$ unknowns (β^j, $j = 1, N1$) which enable us to generate $N1$ equations for satisfying the governing differential equations at the $N1$ boundary nodes which was not possible in the case of direct DRBF method. It should be noted also that for the special case of $L =$ Laplace operator, $L(\Psi) = 0$ and therefore the above equations are decoupled so that α and β can be determined independently.

3. EXTENSION TO NON-LINEAR PROBLEMS

If Eqs. (1a) and (1b) take the following forms,

$$
L(\nabla^2 u, \nabla u, u) = Nf(\nabla u, u, x) \tag{7a}
$$

$$
B\left(u, \frac{\partial u}{\partial n}\right) = Ng(\nabla u, u, x) \tag{7b}
$$

where Nf and Ng arenonlinear functions of their arguments, then Eq. (6) becomes

$$
\begin{bmatrix} L(\Phi) & 0 \\ B(\Phi) & B(\Psi) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} Nf(\nabla u, u, x) \\ Ng(\nabla u, u, x) \end{bmatrix} . \tag{8}
$$

For problems involving high gradients, the solution of the above nonlinear equations cannot be obtained by direct iterative procedures and therefore, the following incremental-iterative procedure is suggested:

1. Set $Nf = Ng = 0$, apply a small increment of the boundary condition and perform the following iterative steps:

   (a) Solve Eq. (8) for the initial estimates of coefficients α and β.

   (b) Use Eq. (5) and its normal derivative to compute the initial estimates for u, $\mathrm{d}u/\mathrm{d}n$, respectively, on the boundary and u, Δu inside the domain.

   (c) Use the values obtained above to compute the initial estimate of Nf and Ng.

   (d) Repeat steps (a) to (c) until convergence is achieved. If the iterations do not converge, reduce the increment of boundary conditions and repeat the above procedure until convergence.

2. Apply another increment and use the all the values obtained at the end of the previous incremental step as estimates for the unknowns and repeat the iterative steps explained above

3. Add more increments while repeating the above iterative procedure until the total boundary conditions are applied.

4. NUMERICAL EXAMPLES

The following examples are given to show the accuracy of the proposed MRBF method. In all examples, the solution of the MRBF method ($\Phi = R^3$ coupled with $\Psi = \ln(R)$) is compared with the DRBF method by employing the two RBFs $\Phi = R^3$ and $\Phi = (c + R^2)^{\frac{1}{2}}$ in order to show the advantage of adding the logarithmic radial basis function in capturing the values of high gradients at the boundary. The shape factor c was chosen by numerical experiments which have shown that more accurate results are obtained when the value of c ranges from 0.1 to 0.3. For the MRBF method, the nodes representing the Logarithmic RBF (Ψ) are equal in number to the boundary nodes and located at a distance $d = 0.5$ from the boundary.

Example 1: Diffusion reaction problem-linear case

The governing equation is given by

$$\nabla^2 u = k u^n. \tag{9}$$

The domain of the problem is assumed to be a circle of radius $r = 1$ subjected to a Dirichlet boundary condition of $u = 1$. Although, the problem is axi-symmetric and therefore can be solved as one-dimensional, the analysis here is performed by treating the problem as two-dimensional using rectangular coordinates x_1 and x_2. Two types of grids are used to distribute the domain and boundary nodes. The first one consists of 110 uniformly distributed nodes with 11 nodes in the radial direction and 10 nodes in the circumferential direction. The second grid has the same number of nodes but the nodes in the radial direction are non-uniformly distributed so that the size of the grid decreases at the neighborhood of the boundary in order to capture the singularity of the gradient at the boundary. The linear problem, where $n = 1$ and $k = 1$, 10, and 100, is considered first. Using the uniform mesh, the solution of the concentration u is obtained as shown in Fig. 1 which indicates that both RBF and MRBF are in good agreement with the exact solution for all values of k. The solution for the gradient $\mathrm{d}u/\mathrm{d}r$ is given in Figs. 2, 3 and 4 for $k = 1$,10 and 100 respectively. For $k = 1$ and $k = 2$ both RBF and MRBF solutions are in good agreement with the exact solution. However, for $k = 100$, the DRBF solutions for the gradient deviate from the exact solution at the boundary as shown in Fig. 4. The same figure clearly shows that the MRBF solution for the gradient is in a very good agreement with the exact solution for $k = 100$ and is able to capture the singularity at the boundary. In order to improve the DRBF results of the gradient for $k = 100$, the analysis is performed using the non-uniform grid and the results are plotted in Fig. 5. Although, the use of the non-uniform grid improved the DRBF results somewhat, the accuracy of the proposed MRBF is still better.
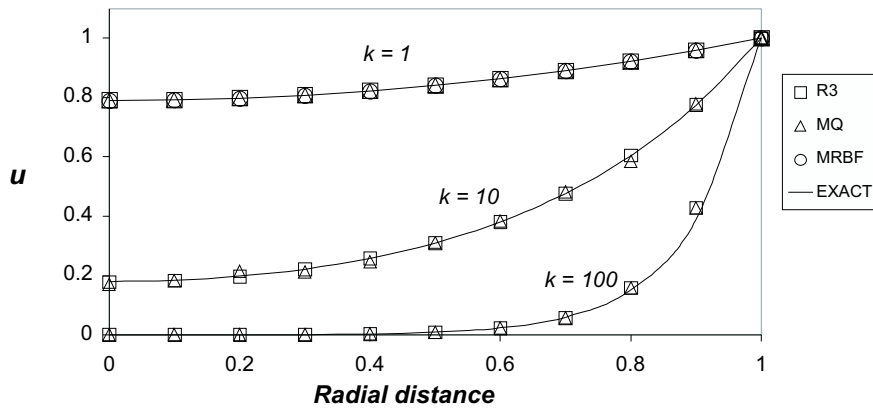
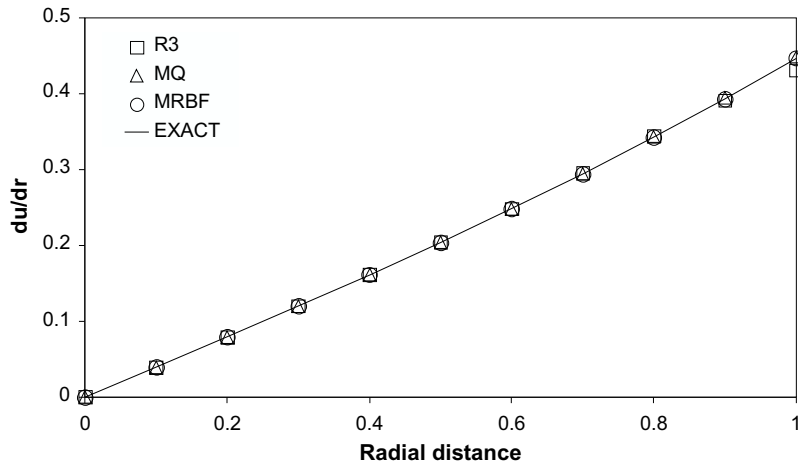**Fig. 1.** Variation of u along the radial direction (Example 1)
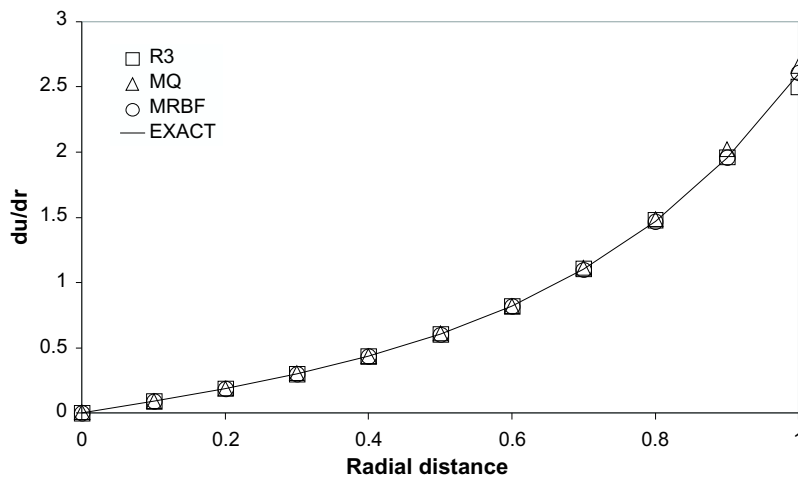

Fig. 2. Variation of $\mathrm{d}u/\mathrm{d}r$ ($k = 1$, Example 1)

Fig. 3. Variation of $\mathrm{d}u/\mathrm{d}r$ ($k = 10$, Example 1)
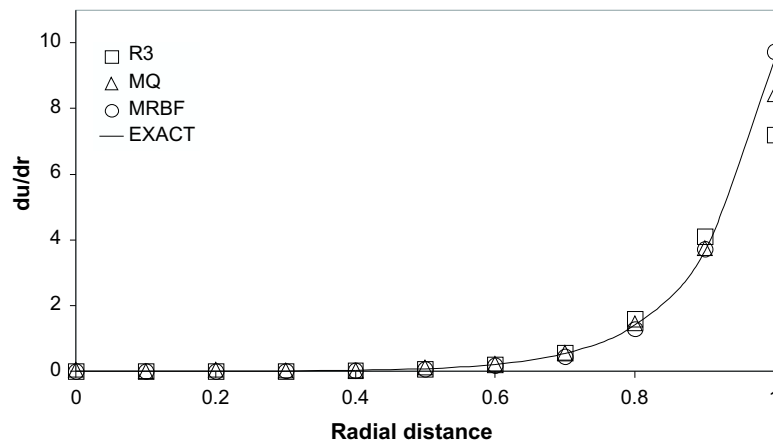
Fig. 4. Variation of $\mathrm{d}u/\mathrm{d}r$ ($k = 100$, Example 1)
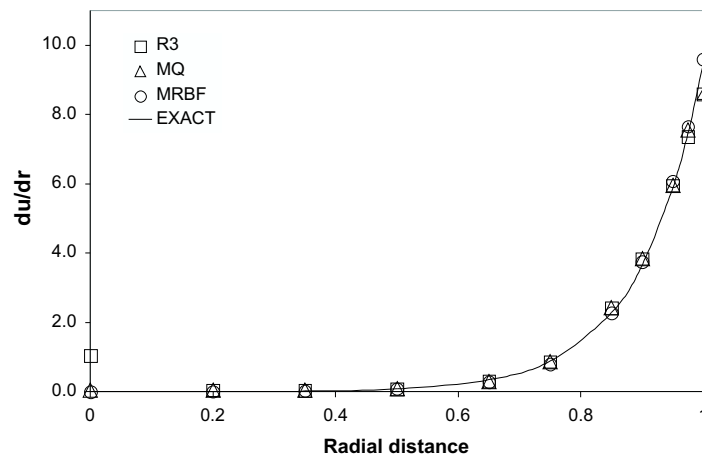

Fig. 5. Variation of $\mathrm{d}u/\mathrm{d}r$ using non-uniform grid ($k = 100$, Example 1)

Example 2: Diffusion reaction problem-nonlinear case

Next, let us reconsider the non-linear case of the problem in Example 1, where $k = 4(x_1^2 + x_2^2 + a^2)$ and $n = 3$ for the same geometry but with the boundary condition $u = -1/(1 - a^2)$. The analytical solution is $u = -1/(x_1^2 + x_2^2 - a^2)$ which can be verified by substituting in the governing differential equation. The singular behavior of the solution at the boundary depends on the given constant a and therefore, in order to asses the capability of the two methods in capturing the solution singularity at the boundary, a is assumed to take the two values of 2 and 1.1. The solutions of u and $\mathrm{d}u/\mathrm{d}r$ for $a = 2$ are given in Figs. 6 and 7 respectively. The two figures show very good agreement among all methods for all points except at the boundary point at which the gradient solution of DRBF (using R^3) deviates slightly from the exact solution. The results for $a = 1.1$ are given in Figs. 8 and 9 for u and $\mathrm{d}u/\mathrm{d}r$, respectively. Figure 8 shows clear deviations of both DRBF solutions (R^3 and MQ) from the exact solution at all points except the boundary point at which all solutions are forced to satisfy the boundary condition. On the other hand, the proposed MRBF method is in a very good agreement with the exact solution at all points. Figure 9 shows that both DRBF solutions for the gradient at the boundary deviate substantially from the exact solution while the proposed MRBF method is in good agreement with the exact solution. In order to improve the DRBF results of the gradient for $a = 1.1$, the analysis is performed using the non-uniform grid and the results are plotted in Figs. 10 and 11 for u and $\mathrm{d}u/\mathrm{d}r$ respectively. The figures show an improvement in the solution of DRBF (R^3) for u only. Furthermore, both DRBF methods (R^3 and MQ) are not able
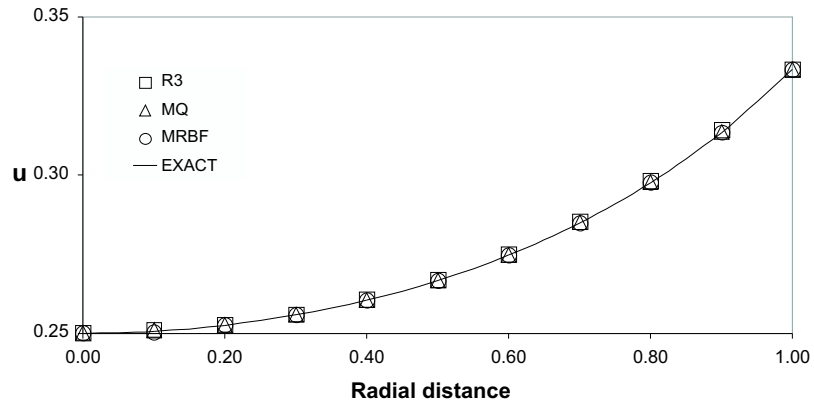
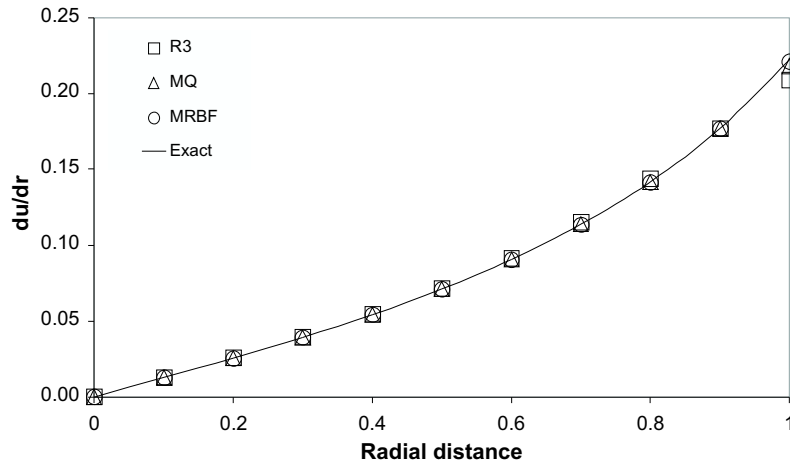**Fig. 6.** Variation of u along the radial direction ($a = 2$, Example 2)

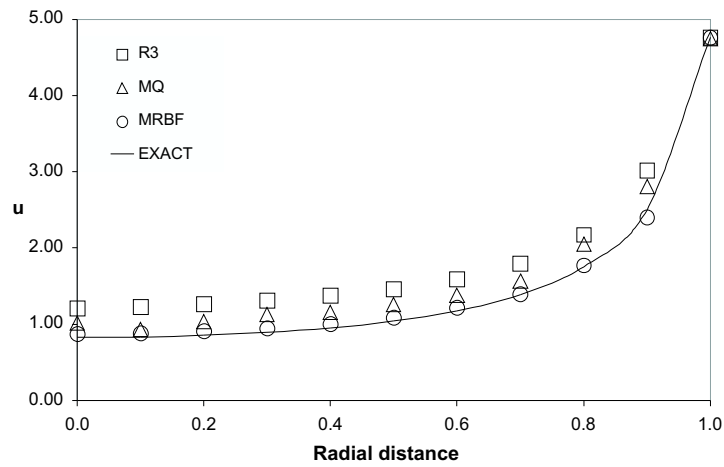
Fig. 7. Variation of du/dr ($a = 2$, Example 2)

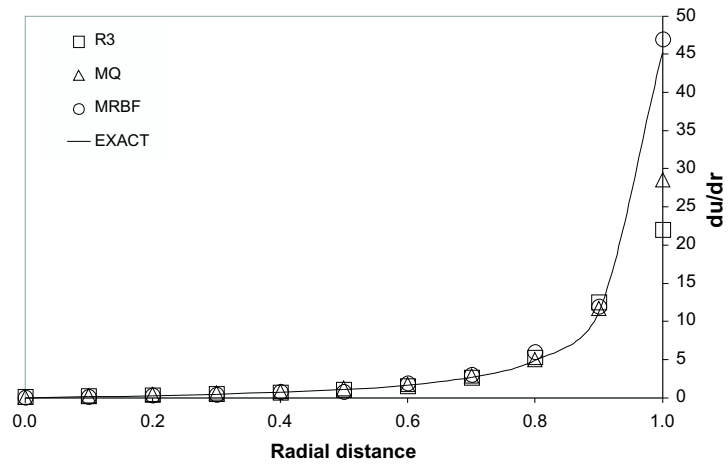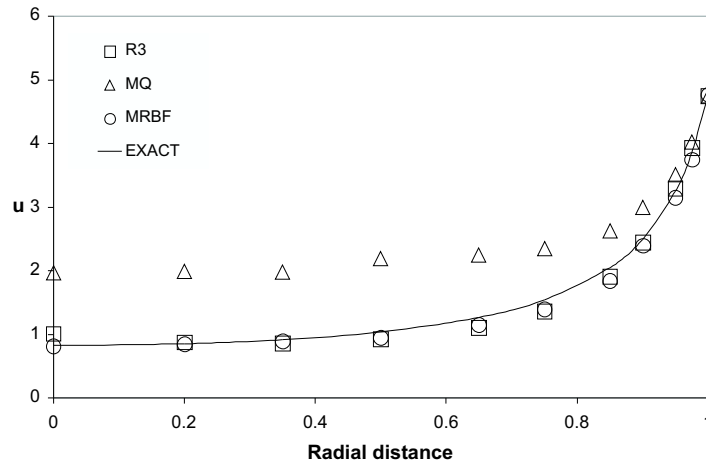**Fig. 8.** Variation of u along the radial direction ($a = 1.1$, Example 2)

Fig. 9. Variation of $\mathrm{d}u/\mathrm{d}r$ ($a = 1.1$, Example 2)

Fig. 10. Variation of u using non-uniform grid ($a = 1.1$, Example 2)
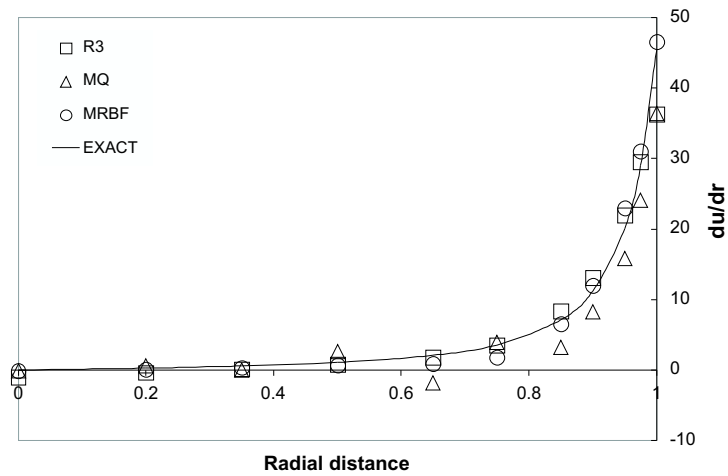
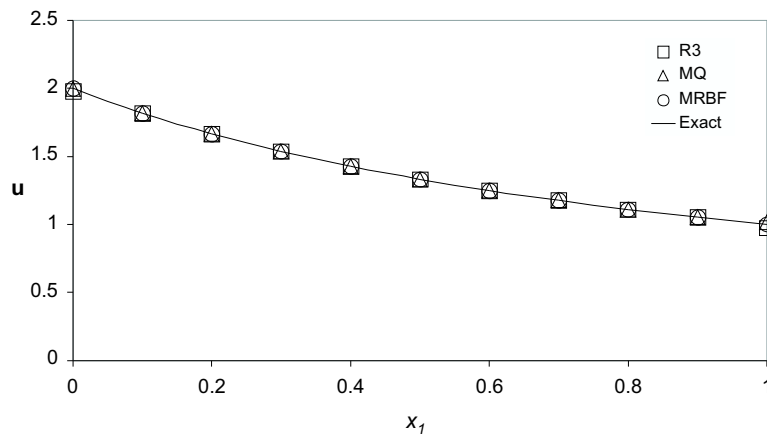**Fig. 11.** Variation of $\mathrm{d}u/\mathrm{d}r$ using non-uniform grid ($a = 1.1$, Example 2)

to capture the singularity of the gradient at the boundary while MRBF is still in good agreement with the exact solution.

Example 3: Burger's equation

This equation is given by

$$\nabla^2 u = -u\,\frac{\partial u}{\partial x_1}\ .\tag{10}$$

The domain of the problem is assumed to be a unit square $\Omega = (0,1) \times (0,1)$ with the Dirichlet boundary condition of $u = 2/(x_1 + a)$ which is also a particular solution for the above equation and therefore represents the exact solution. The values considered for a are 1 and 0.1. The domain is first modeled by a uniform grid of 11×11 nodes. The results for the velocity u are given in Figs. 12 and 13 for $a = 1$ and $a = 0.1$, respectively. The two figures show that both DRBF and MRBF agree well with the exact solution. The results for the gradient $\mathrm{d}u/\mathrm{d}x_1$ are given in Figs. 14 and 15. The two figures show that MRBF results agree very well with the exact solution while DRBF, again, gives considerable errors for the gradient at the boundary $x_1 = 0$, especially for the critical value of $a = 0.1$. In order to improve the accuracy of the DRBF solutions, the analysis is repeated for $a = 0.1$ using a non-uniform grid with decreasing spaces between the nodes at the neighborhood of the boundary $x_1 = 0$. The results of this analysis are given in Fig. 16 which shows some improvement in the DRBF solutions while the MRBF solution is in complete agreement with the exact solution.

Fig. 12. Variation of u along x_1 ($a = 1$, Example 3)
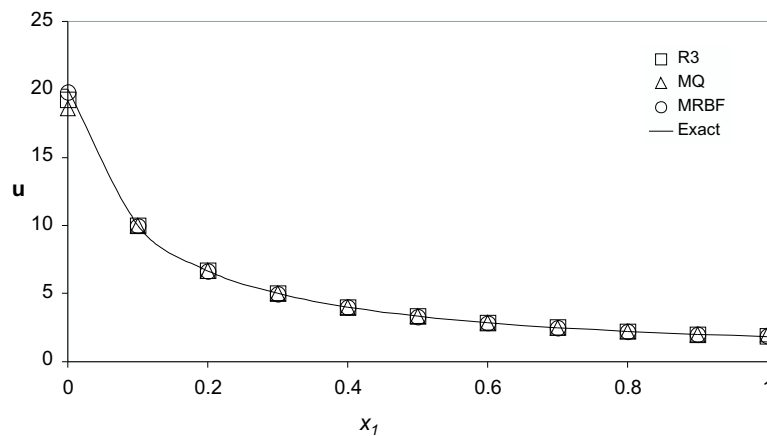
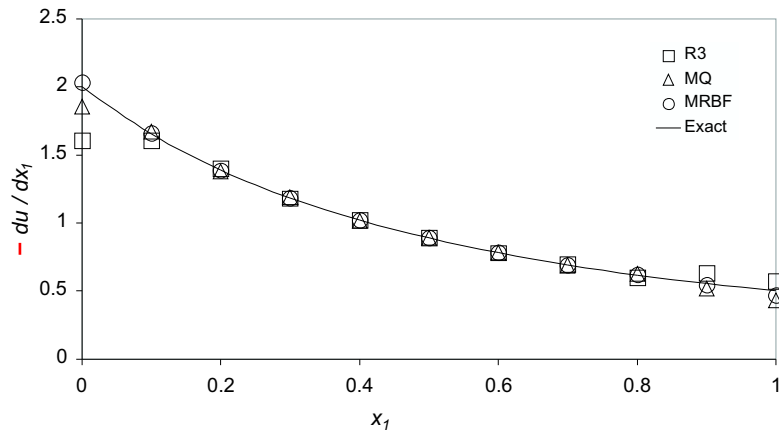**Fig. 13.** Variation of u along x_1 ($a = 0.1$, Example 3)

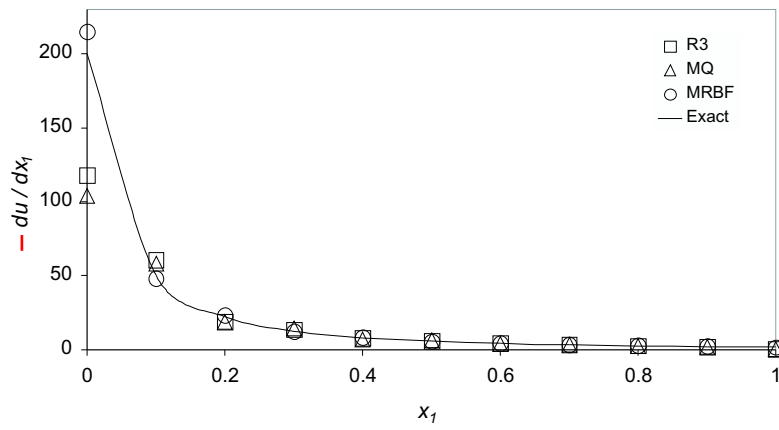**Fig. 14.** Variation of $\mathrm{d}u/\mathrm{d}x_1$ along x_1 ($a = 1$, Example 3)

Fig. 15. Variation of $\mathrm{d}u/\mathrm{d}x_1$ along x_1 ($a = 0.1$, Example 3)

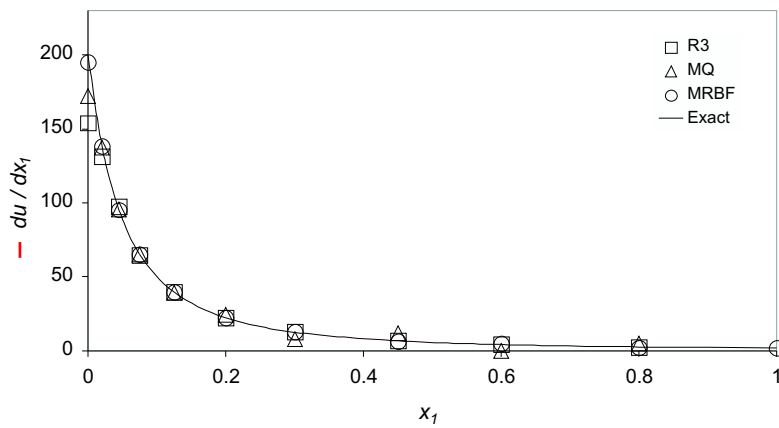**Fig. 16.** Variation of $\mathrm{d}u/\mathrm{d}x_1$ along x_1, using non-uniform grid ($a = 0.1$, Example 3)

5. CONCLUSION

A simple, yet efficient and truly meshless method is presented for solving general linear and non-linear Poisson-type differential equations. The numerical examples show that the addition of the fundamental solution to the standard global radial basis functions used in DRBF method greatly improves its accuracy, especially for problems involving high gradients at the boundary. The only disadvantage of the proposed MRBF method is the determination of a new parameter that defines the distance between the domain boundary and the contour of the knots of the fundamental solution. Although in this study, a wide range of .2 to 1.5 times the largest dimension of the domain of the problem does not affect the accuracy of the results, the issue needs to be investigated for other boundary value problems. Although the formulation and application are given for 2-D Poisson-type differential equations, the procedure is applicable for other types of differential equations provided that a fundamental solution is available for the considered differential equation.

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