# ITERATIVE COUPLING OF BOUNDARY AND FINITE ELEMENT METHODS IN ELASTO-PLASTICITY

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ABSTRACT

The possibility of extending the sequential Dirichlet Neumann boundary element-finite element coupling method to elasto-plasticity is presented in this paper. The successive computation of the displacements and forces/tractions on the interface of the finite element and boundary element sub-domains is performed through an iterative procedure. The procedure is implemented in a computer program, which is tested through numerical examples. The solution of the present method is compared to the conventional finite element method in terms of accuracy and CPU-time.

## 1 INTRODUCTION

The finite element method (FEM) and the boundary element method (BEM) are well known as powerful numerical technique for solving wide range of problems in applied science and engineering. Both the FEM and the BEM have their own range of applications where they are most efficient.

For certain categories of problems, neither the BEM nor the FEM is best suited and it is natural to attempt to couple these two methods in an effort to create a finite element-boundary element method (FEBEM) that combines all their advantages and reduces their disadvantages. Unfortunately the usual derivations for the BEM gives rise to a set of equations, which are not directly compatible with the FEM equations. The systems of equations, produced by the two methods, are expressed in terms of different variables and cannot be linked as they stand. Conventionally coupling the FEM and the BEM is achieved by combining the discretized equations for the BEM and FEM sub-domains, see, i.e., references [1-8] not to mention many others. However, the algorithm for constructing the entire equation is highly complicated. In order to overcome the stated inconvenience, domain decomposition coupling procedures were developed [9-15] where there is no need to combine the coefficient matrix for the FEM and BEM sub-domains. In these coupling algorithms, separate computing for each sub-domain and successive renewal of the variables on the interface of the both sub-domains are performed to reach the final convergence. Gerstle et al. [9] and Perera et al. [10] presented solution schemes, which utilize the conjugate gradient method and the Schur complement, respectively, for the renewal of the unknowns at the interface. Kamiya et al. [11] employed the renewal schemes known as Schwarz Neumann-Neumann and Schwarz Dirichlet-Neumann. Kamiya and Iwase [12] introduced an iterative analysis using conjugate gradient and condensation. Lin et al. [13], and Feng and Owen [14] presented a method, which is considered as a sequential form of the Schwarz Dirichlet-Neumann method. Elleithy and Al-Gahtani [15] presented an overlapping domain decomposition method for coupling of the FEM and BEM. The domain of the original problem is subdivided into a FEM sub-domain, a BEM sub-domain, and a common region, which is modeled by both methods.

The above iterative coupling methods, however, are only limited to linear problems. The objective of this paper is to extend the application of the sequential Schwarz Dirichlet-Neumann iterative coupling method to elasto-plasticity. Applications in infinite domain elasto-plasticity are presented. The conventional FEM computations are also performed, and a critical comparison of the results is made.

## 2 ITERATIVE COUPLING APPROACH IN ELASTO-PLASTICITY

In this section we consider the extension of the sequential Schwarz Dirichlet-Neumann iterative coupling method presented by Lin et al. [13], and Feng and Owen [14] to elasto-plasticity. The domain of the original problem $\Omega$ is decomposed into two sub-domains $\Omega^B$ and $\Omega^F$. Now, let us define the following vectors (Figure 1):

$\{u_B\}$: displacement in the BEM sub-domain,

$\{u_B^I\}$: displacement on the BEM/FEM interface (but it is approached from the BEM sub-domain),

$\{u_B^B\}$: displacement in the BEM sub-domain except $\{u_B^I\}$,

$$\{u_B\} = \{u_B^B, u_B^I\}^T$$

$\{u_F\}$: displacement in the FEM sub-domain,

$\{u_F^I\}$: displacement on the BEM/FEM interface (approached from the FEM sub-domain), and

$\{u_F^F\}$: displacement in the FEM sub-domain except $\{u_F^I\}$,

$$\{u_F\} = \{u_F^F, u_F^I\}^T$$

Similarly, one can denote the BEM traction by $t_B^B$ and $t_B^I$ and FEM force vectors by $f_F^F$ and $f_F^I$. Disregarding body forces, the assembled boundary element equations for an elastic region are given by:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_B^B \\ u_B^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} t_B^B \\ t_B^I \end{Bmatrix} \tag{1}$$

For an elasto-plastic analysis, the incremental form of the FEM equations can be written as:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \Delta u_F^F \\ \Delta u_F^I \end{Bmatrix} = \begin{Bmatrix} \Delta f_F^F \\ \Delta f_F^I \end{Bmatrix} \tag{2}$$

It should be noted that for each load increment, Equations (2) are nonlinear and therefore are solved iteratively. At the interface, the compatibility and equilibrium conditions should be satisfied, i.e.,

$$\{u_B^I\} = \{u_F^I\} \quad \in \Gamma^I \tag{3}$$

$$\{f_F^I\} + [M]\{t_B^I\} = 0 \quad \in \Gamma^I \tag{4}$$

where, $[M]$ is the converting matrix due to the weighing of the boundary tractions by the interpolation functions on the interface. The iterative coupling method can be summarized as follows:

1. Given the initial guess $\{u_{B,0}^{I}\}=\{\overline{u}\}$.
2. For $n = 0, 1, 2, \ldots\ldots$, do until convergence

   Solve Equation (1) and get $\{t_{B,n}^{I}\}$

   Solve Equation (4) and obtain $\{f_{F,n}^{I}\}$

   For the FEM region

   For $i = 1, 2, \ldots\ldots$, specified number of increments

   Solve Equation (2) for $\{\Delta u_{F,i}^{I}\}_n$

   Apply $\{u_{F,i+1}^{I}\}_n = \{u_{F,i}^{I}\}_n + \{\Delta u_{F,i}^{I}\}_n$

   Obtain $\{u_{F,n}^{I}\}$

   Apply $\{u_{B,\,n+1}^{I}\}=(1-\alpha)\{u_{B,\,n}^{I}\}+\alpha\,\{u_{F,\,n}^{I}\}$

   where $\alpha$ is a relaxation parameter to accelerate and/or speed up convergence.

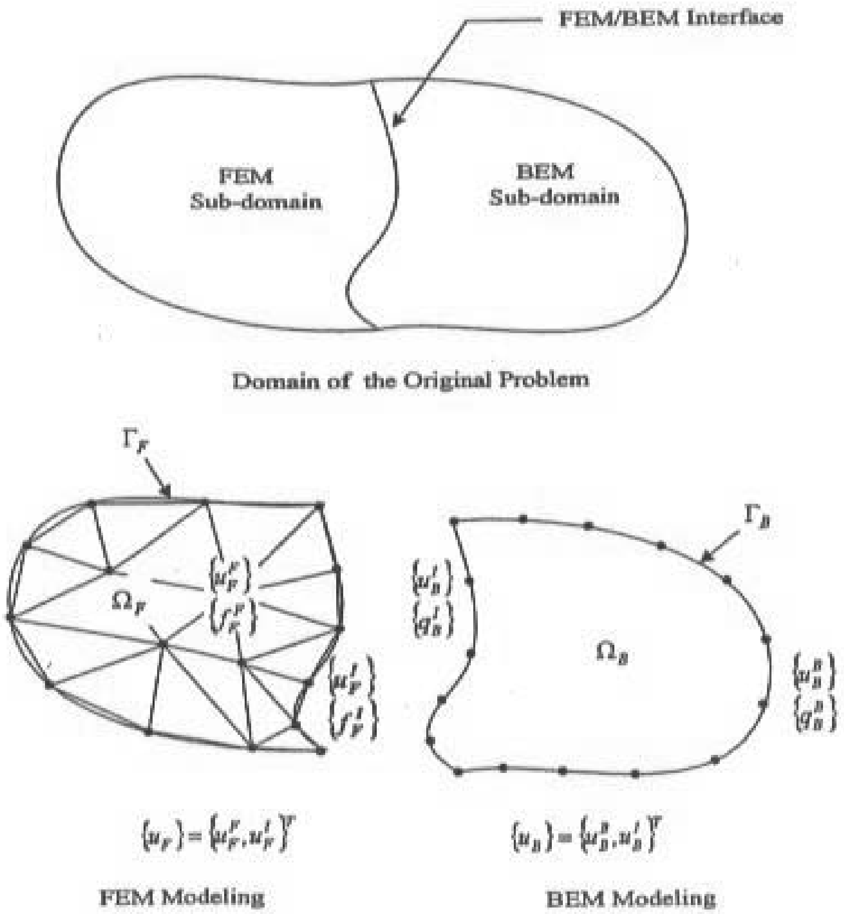


Figure 1: Domain Decomposition

## 3 APPLICATIONS

Consider the excavation of a circular tunnel opening in a geological medium. The tunnel is deeply inserted in an intact rock. The soil is governed by the true Drucker-Prager yield criterion without any hardening effect. The plane strain condition is assumed to prevail. The radius of the tunnel $R$ is taken as 100 units. The material properties employed are as follows: Young's modulus E=$2.1\text{x}10^4$ units, Poisson's ratio $\nu = 0.18$, Cohesion $c = 10$ units and angle of internal friction $\phi = 41^o$.

### 3.1 Elastic Analysis

The purpose of this analysis is to compare the results of the present method with the available exact solution. The stress condition in the geological medium is assumed to be hydrostatic and the stress is taken to be 10 units. First the excavation of tunnel is analyzed with the FEM. In this case the infinite domain is truncated at 4.3, 8.7 and 15 times the radius of the tunnel from the center of the tunnel opening. At the boundary of the tunnel, the forces corresponding to in-situ state of stress condition are computed at the nodal points and applied in the opposite direction to simulate the excavation of the opening. The problem is then analyzed with the FEBEM. The FEBEM interface is set at 3.6 times the radius of the tunnel. Figure 2 shows the discretization with the FEM and FEBEM. Table 1 shows the number of elements and CPU-time required for the analysis with FEM and FEBEM. The difference in the CPU-time required for analysis is considered to be marginal.

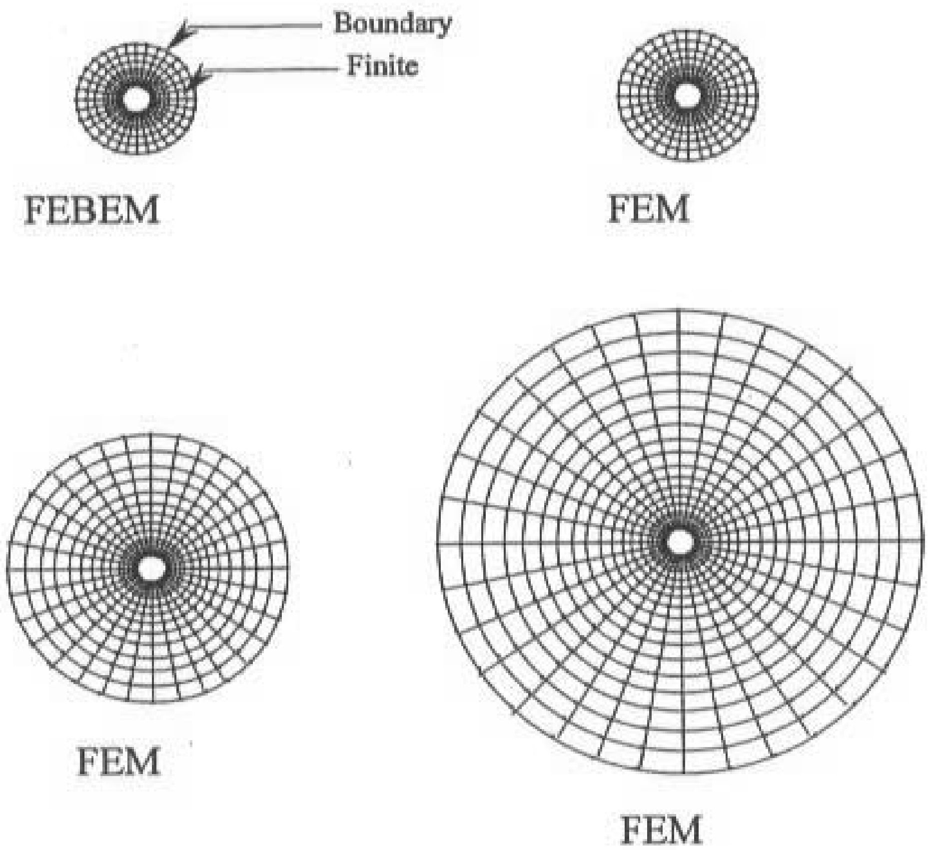


Figure 2: FEM and FEBEM Discretization for the Tunnel

Table 1: Comparison of Input Data and CPU-Time for Elastic Analysis of the Tunnel

| Method | Boundary Distance | Number of Finite Elements | Number of Boundary Elements | CPU-time in Sec. |
|---|---|---|---|---|
| FEM | 4.3 R | 288 | - | 2 |
| FEM | 8.7 R | 448 | - | 3 |
| FEM | 15 R | 608 | - | 4 |
| FEBEM | 3.6 R | 256 | 32 | 3 |

Figure 3 shows the radial displacements by the FEM and FEBEM as compared to closed form solution. It may be observed that as the extent of boundary distance for the FEM increases, better accuracy is achieved. Table 2 shows the displacement at the boundary of the tunnel $u_R$ and the percentage error compared to exact solution. The results clearly show the advantage of using the FEBEM in terms of accuracy. Another advantage, which cannot be seen from the results, is the incredible reduction of data preparation required for the FEBEM analysis as compared with the FEM.

Table 2: Displacement at the Boundary of the Tunnel for Elastic Analysis

| Method | Boundary Distance | $\frac{u_R}{R}$ | % Error |
|---|---|---|---|
| Exact | - | $0.0562 \times 10^{-2}$ | - |
| FEM | 4.3 R | $0.0488 \times 10^{-2}$ | 13.17 |
| FEM | 8.7 R | $0.0542 \times 10^{-2}$ | 3.56 |
| FEM | 15 R | $0.0554 \times 10^{-2}$ | 1.42 |
| FEBEM | 3.6 R | $0.0565 \times 10^{-2}$ | 0.53 |

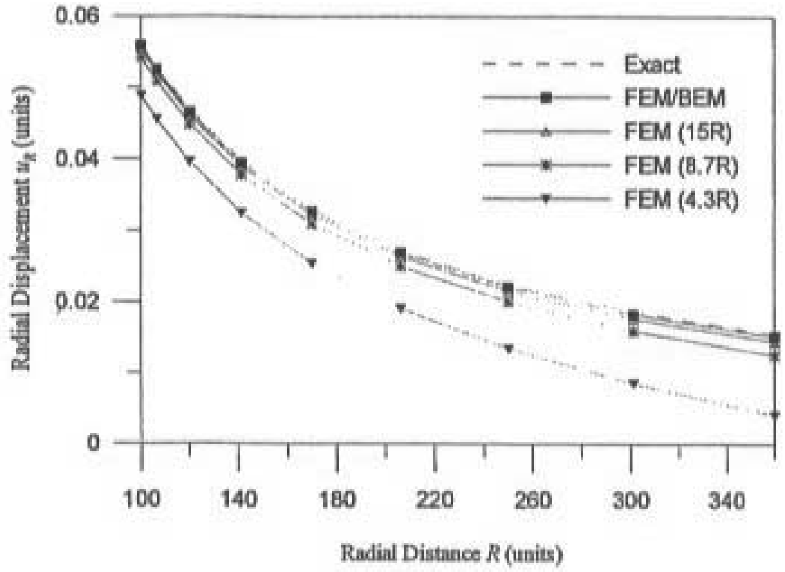


Figure 3: Radial Displacement for Elastic Analysis of the Tunnel

### 3.2 Elasto-Plastic Analysis

The elasto-plastic analysis has been used to study the excavation of the tunnel. The excavation has been simulated in a single stage. At the boundary of the tunnel, the forces corresponding to in-situ state of stress condition (i.e. hydrostatic stress is taken as 25 units) were computed at the nodal points and applied in the opposite direction to simulate the excavation of the tunnel. Again the problem is analyzed with the FEM and FEBEM using the discretization shown in Figure 2. The radial displacements for both methods are shown in Figure 4. It is observed that as the extent of boundary distance for the FEM increases, the FEM converges to the FEBEM solution. The yielded zones determined by all methods are identical and are obtained as 70 units from the boundary of the tunnel.

Table 3 gives a comparison between the FEM and FEBEM in terms of the computation time required for convergence. The CPU-time is least for the FEM with a boundary distance of $4.3R$ and highest for the FEM with $15R$. In order to fairly, compare the two methods in terms of CPU-time, one should consider the accuracy. Although, the FEM yields less CPU-time when the boundary is truncated at or less than $8.7R$ its solution is not accurate as given in Figure 4. However, the FEM solution with boundary truncated at $15R$ has the same level of accuracy as that by FEBEM, but it requires a higher CPU-time.

Table 3: Comparison of CPU-Time for Elasto-Plastic Analysis of the Tunnel

| Method | Boundary Distance | CPU-time (Sec.) |
|---|---|---|
| FEM | 4.3 R | 4 |
| FEM | 8.7 R | 5 |
| FEM | 15 R | 8 |
| FEBEM | 3.6 R | 6 |

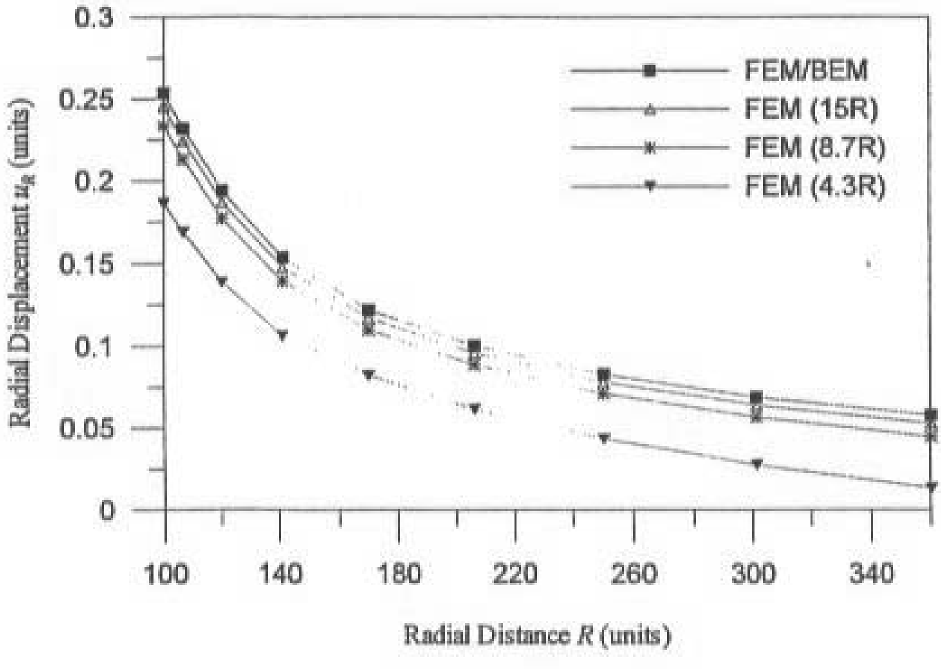


Figure 4: Radial Displacements for Elasto-Plastic Analysis of the Tunnel

## 4. CONCLUSIONS

The extension of the sequential Dirichlet Neumann Iterative coupling of FEM and BEM to elasto-plasticity is investigated in this paper. An application in infinite elasto-plastic domain problem is presented. Beside the convenience of less input data, the iterative FEBEM has the advantage of preserving the identity of both FEM and BEM. The numerical examples show that the iterative FEBEM, in general, yields more accurate results and less CPU-time as compared to FEM.

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