EXACT STIFFNESSES FOR TAPERED MEMBERS

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ABSTRACT: A simple method for deriving closed-form expressions for the components of the stiffness matrix and fixed-end forces and moments for tapered members is presented. The governing differential equations and the boundary integral method are used to obtain exact expressions for axial, torsional, and flexural stiffnesses. The necessary fixed-end forces and moments are also derived. The procedure of the proposed method is explained through a practical class of tapered members. The procedure, however, can be extended to other axial, torsional, and flexural stiffness variations. The correctness of the obtained stiffness expressions is verified through numerical examples.

INTRODUCTION

Members with variable cross sections are used in many structural applications to optimize the distribution of weight and strength and sometimes to satisfy architectural and functional requirements. Examples are highway bridges, buildings, space and aircraft structures, as well as many mechanical components and machines. Since these members are involved in many important structures, it is necessary to analyze them with a greater precision.

The analysis of tapered members are covered in many classical texts on structural analysis, e.g., Timoshenko and Young (1965) and Hibbeler (1990). The analysis involves lengthy calculations and requires tables and charts which are not applicable in general cases. The other alternative is to use numerical methods such as the finite-element method, e.g., Bathe (1982), where the member is represented by a number of segments and the stiffness matrices for the segments are superimposed to produce the stiffness matrix for the whole member. The increase in the number of equations due to the process of member discretization is not disadvantageous anymore, because of the emergence of modern super computers. The real disadvantage is the huge amount of input data required, especially in the case of large structures.

The purpose of this paper is to present a simple, yet exact, procedure for deriving closed-form stiffness expressions for symmetrically tapered members (having a straight centroidal axis). The proposed procedure is based on the boundary integral method which does not require discretization of the member. The method produces exact relations among the forces and displacements at the member ends. The equilibrium equations of the member are based on the Bernolli-Euler beam theory, which is quite adequate provided the tapering ratio is not too great [see Boley (1963)]. The procedure is explained through a practical class of tapered members. Finally, the correctness of the derived expressions is demonstrated through numerical examples.

PROBLEM STATEMENT

Consider a nonuniform Bernoulli-Euler beam of length L as shown in Fig. 1. The static stiffness equation can be written as

$$\mathbf{F} = \mathbf{K}\mathbf{U} \tag{1}$$

where $\mathbf{K}$ = a 12×12 stiffness matrix; $\mathbf{F}$ = a column matrix containing the axial forces (F_1 and F_7), torques (F_4 and F_{10}), shear forces, and bending moments associated with flexure in the xy-plane (F_2, F_8, F_6, and F_{12}), and the shear forces and bending moments associated with flexure in the xz-plane (F_3, F_9, F_5, and F_{11}); and $\mathbf{U}$ = a column matrix containing the corresponding axial, torsional, and flexural displacements. It is assumed that: (1) the beam is made of a homogeneous isotropic linear elastic material with elastic modulus E and shear modulus G; (2) the centroidal axis (x-axis) is straight and the directions of the principal axes (y- and z-axes) are the same for all cross sections; and (3) warping and coupling between torsional and flexural behaviors are negligible. In addition to the foregoing, the taper is assumed such that the cross-sectional area $A(x)$, polar moment of inertia $J(x)$, and moment of inertia $I(x)$ are given by

$$A(x) = A_0\left(1 + \frac{cx}{L}\right)^n; \quad J(x) = J_0\left(1 + \frac{cx}{L}\right)^{n+2} \tag{2a,b}$$

and

$$I(x) = I_0\left(1 + \frac{cx}{L}\right)^{n+2} \tag{2c}$$

where A_0, J_0, and I_0 = cross-section properties at the origin; c = a constant that accounts for the degree of tapering; and n = a real number which depends on the shape of the cross section. The applicability of (2) with $n = 1$ and $n = 2$ for different shapes of the cross section are discussed by Banerjee and Williams (1985). The formulation given in the following section, however, is valid for other values of n as well as for other functions $A(x)$, $J(x)$, and $I(x)$.

To derive the stiffness matrix, the differential equations governing the axial, torsional, and flexural behaviors of the beam will be considered separately.

Axial Stiffness

The differential equation is given by

$$\frac{d}{dx}\left[EA(x)\frac{du}{dx}\right] + p(x) = 0, \; x \in (0, L) \tag{3}$$

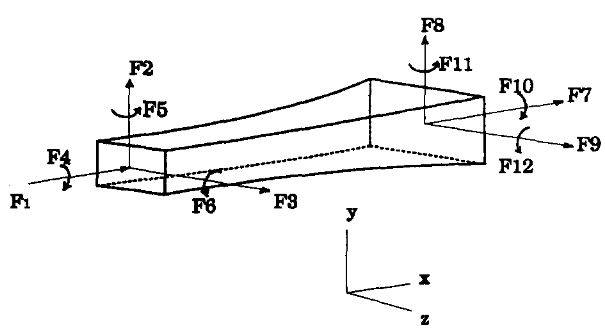

FIG. 1. Sign Convention for Tapered Beam

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Note. Associate Editor: Eric M. Lui. Discussion open until March 1, 1997. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on October 25, 1995. This paper is part of the ***Journal of Structural Engineering***, Vol. 122, No. 10, October, 1996. ©ASCE, ISSN 0733-9445/96/0010-1234–1239/$4.00 + $.50 per page. Paper No. 11886.

where u = axial displacement; and p = distributed axial force. The boundary conditions are

$$u(0) = U_1 \text{ or } EA\frac{du}{dx}(0) = -F_1 \tag{4a}$$

$$u(L) = U_7 \text{ or } EA\frac{du}{dx}(L) = F_7 \tag{4b}$$

To obtain the so-called inverse formulation, multiply both sides of (3) by a weighing function u^* and integrate over the length of the beam twice to get

$$\int_0^L u\frac{d}{dx}\left[EA(x)\frac{du^*}{dx}\right]dx + \left[u^*EA(x)\frac{du}{dx}\right]_0^L - \left[uEA(x)\frac{du^*}{dx}\right]_0^L + \int_0^L u^*p(x)\,dx = 0 \tag{5}$$

Let us choose u^* to be the fundamental solution, i.e., the solution of the following differential:

$$\frac{d}{dx}\left[EA(x)\frac{du^*}{dx}\right] + \Delta(x-\xi) = 0,\ x \in (-\infty, \infty) \tag{6}$$

where $\Delta(x - \xi)$ = Dirac delta function representing a unit axial load applied at the point $x = \xi$. Using (6) in (5), the following is obtained for $0 < \xi < L$:

$$u(\xi) = EA\frac{du^*}{dx}(0,\xi)U_1 - EA\frac{du^*}{dx}(L,\xi)U_7 + u^*(0,\xi)F_1 + u^*(L,\xi)F_7 + \int_0^L u^*(x,\xi)p(x)\,dx \tag{7}$$

Assuming that we can solve (6) for $u^*(x, \xi)$, (7) gives the axial displacement at any point ξ in terms of the boundary values $u(0)$, $u(L)$, $EA\ du/dx(0)$, and $EA\ du/dx(L)$. Applying (7) at the beam ends (i.e., $\xi = 0$ and $\xi = L$), yields the following written in a matrix form:

$$\begin{bmatrix} 1 - EA\dfrac{du^*}{dx}(0,0) & EA\dfrac{du^*}{dx}(L,0) \\ -EA\dfrac{du^*}{dx}(0,L) & 1 + EA\dfrac{du^*}{dx}(L,L)\end{bmatrix}\begin{Bmatrix}U_1\\U_7\end{Bmatrix} = \begin{bmatrix}u^*(0,0) & u^*(L,0)\\ u^*(0,L) & u^*(L,L)\end{bmatrix}\begin{Bmatrix}F_1\\F_7\end{Bmatrix} + \begin{Bmatrix}\displaystyle\int_0^L u^*(x,0)p(x)\,dx\\ \displaystyle\int_0^L u^*(x,L)p(x)\,dx\end{Bmatrix} \tag{8}$$

An integration of (6) yields

$$EA\frac{du^*}{dx} = -\frac{1}{2}\,\mathrm{sgn}(x-\xi) \tag{9}$$

where $\mathrm{sgn}(x - \xi) = 1$ for $x > \xi$ and $\mathrm{sgn}(x - \xi) = -1$ for $x < \xi$. To obtain u^*, integrate (9) over the range ξ to x, i.e.

$$u^*(x,\xi) = -\frac{1}{2}\,\mathrm{sgn}(x-\xi)\int_\xi^x \frac{dx}{EA(x)} \tag{10}$$

Substituting $A(x)$ from (2a) we get

$$u^*(x,\xi) = -\mathrm{sgn}(x-\xi)\frac{L}{EA_0}\frac{1}{2c}\log\left(\frac{L+cx}{L+c\xi}\right),\ n = 1 \tag{11a}$$

$$u^*(x,\xi) = \mathrm{sgn}(x-\xi)\frac{L}{2c(n-1)EA_0}\left[\frac{1}{\left(1+c\dfrac{x}{L}\right)^{n-1}} - \frac{1}{\left(1+c\dfrac{\xi}{L}\right)^{n-1}}\right],\ n > 1 \tag{11b}$$

If the fundamental solutions given by (11) are evaluated by taking the limits as $\xi \to 0$ and $\xi \to L$, (8) becomes

$$\begin{bmatrix}\dfrac{1}{2} & -\dfrac{1}{2}\\ -\dfrac{1}{2} & \dfrac{1}{2}\end{bmatrix}\begin{Bmatrix}U_1\\U_7\end{Bmatrix} = \begin{bmatrix}0 & \alpha\\ \alpha & 0\end{bmatrix}\begin{Bmatrix}F_1\\F_7\end{Bmatrix} + \begin{Bmatrix}\displaystyle\int_0^L u^*(x,0)p(x)\,dx\\ \displaystyle\int_0^L u^*(x,L)p(x)\,dx\end{Bmatrix} \tag{12}$$

where

$$\alpha = -\frac{L}{EA_0}\frac{\log(1+c)}{2c},\ n = 1 \tag{13a}$$

$$\alpha = \frac{L}{EA_0}\frac{1-(1+c)^{1-n}}{2c(1-n)},\ n > 1 \tag{13b}$$

To obtain the stiffness equation, multiply both sides of (12) by the inverse of the first matrix on its right-hand side to get

$$\begin{bmatrix}K_{1,1} & K_{1,7}\\ \text{symm} & K_{7,7}\end{bmatrix}\begin{Bmatrix}U_1\\U_7\end{Bmatrix} = \begin{Bmatrix}F_1\\F_7\end{Bmatrix} + \begin{Bmatrix}P_1\\P_7\end{Bmatrix} \tag{14}$$

where

$$K_{1,1} = K_{7,7} = \frac{EA_0}{L}\frac{c}{\log(1+c)},\ n = 1;\ K_{1,7} = K_{7,1} = -K_{1,1},\ n = 1 \tag{15a,b}$$

$$K_{1,1} = K_{7,7} = \frac{EA_0}{L}\frac{c(1-n)}{[(1+c)^{1-n} - 1]},\ n > 1 \tag{15c}$$

$$K_{1,7} = K_{7,1} = -K_{1,1},\ n > 1 \tag{15d}$$

and the column matrix $\{P\}$ contains the fixed-end axial forces which are given by

$$\begin{Bmatrix}P_1\\P_7\end{Bmatrix} = \begin{bmatrix}0 & \dfrac{1}{\alpha}\\ \dfrac{1}{\alpha} & 0\end{bmatrix}\begin{Bmatrix}\displaystyle\int_0^L u^*(x,0)p(x)\,dx\\ \displaystyle\int_0^L u^*(x,L)p(x)\,dx\end{Bmatrix} \tag{16}$$

The components of this matrix depend on the applied load $p(x)$ as will be discussed later.

Torsional Stiffness

For circular cross sections, the differential equation is given by

$$\frac{d}{dx}\left[GJ(x)\frac{d\theta}{dx}\right] + t(x) = 0,\ x \in (0, L) \tag{17}$$

where θ = torsional rotation; and t = applied distributed torque. Eq. (17) can also be used to approximate the torsional behavior of several cross sections if G is given a suitable artificial value. Since (17) is similar to (3), and since the function $GJ(x)$ is two orders higher than $EA(x)$ [see (2)], we can use the results of (15b), after replacing n by $n + 2$, i.e., the torsional stiffness equation becomes

$$\begin{bmatrix}K_{4,4} & K_{4,10}\\ K_{10,4} & K_{10,10}\end{bmatrix}\begin{Bmatrix}U_4\\U_{10}\end{Bmatrix} = \begin{Bmatrix}F_4\\F_{10}\end{Bmatrix} + \begin{Bmatrix}P_4\\P_{10}\end{Bmatrix} \tag{18a}$$

where

$$K_{4,4} = K_{10,10} = \frac{GJ_0}{L} \frac{-(1+n)c}{[(1+c)^{-(1+n)} - 1]} \quad (18b)$$

$$K_{4,10} = K_{10,4} = -K_{4,4}, \; n \geq 1 \quad (18c)$$

and P_4, P_{10} = fixed-end torques which will be discussed later.

Flexural Stiffness

The differential equation governing flexure in the xy-plane is given by

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right] + q(x) = 0, \; x \in (0, L) \quad (19)$$

where $w(x)$ = lateral deflection; and q = distributed lateral load. The slope $S(x)$, moment $M(x)$, and shear $V(x)$ are related to the primary variable $w(x)$ by

$$S(x) = \frac{dw}{dx}, \; M(x) = EI(x)\frac{d^2w}{dx^2}; \text{ and } V(x) = \frac{d}{dx}\left[EI(x)\frac{d^2w}{dx^2}\right] \quad (20a,b)$$

The end conditions are $w = U_2$, $S = U_6$, $M = -F_6$, $V = F_2$ at $x = 0$; and $w = U_8$, $S = U_{12}$, $M = F_{12}$, $V = -F_8$ at $x = L$. Following the same procedure, i.e., multiplying (19) by a fundamental solution w^* and integrating four times, the following equation, which is similar to (7), can be obtained:

$$w(\xi) = [w^*(x, \xi)V(x) - S^*(x, \xi)M(x) + M^*(x, \xi)S(x) - V^*(x, \xi)w(x)]_{x=0}^{x=L} + \int_0^L q(x)w^*(x, \xi)\,dx \quad (21)$$

where S^*, M^*, V^* are related to w^* through the same differential operators given by (20). Once the fundamental solution w^* is known, (21) gives the deflection at any point inside the beam in terms of the values of deflection, slope, moment, and shear at the beam ends and the given load $q(x)$. An equation for the slope can be obtained by differentiating (21) with respect to ξ. The result is

$$S(\xi) = [w^{*\prime}(x, \xi)V(x) - S^{*\prime}(x, \xi)M(x) + M^{*\prime}(x, \xi)S(x) - V^{*\prime}(x, \xi)w(x)]_{x=0}^{x=L} + \int_0^L q(x)w^{*\prime}(x, \xi)\,dx \quad (22)$$

where $'$ indicates differentiation with respect to ξ. Applying (21) and (22) at the beam ends and using the boundary conditions, the following are obtained:

$$\begin{bmatrix} 1 - v^*(0,0) & M^*(0,0) & V^*(L,0) & -M^*(L,0) \\ -v^{*\prime}(0,0) & 1 + M^{*\prime}(0,0) & V^{*\prime}(L,0) & -M^{*\prime}(L,0) \\ -v^*(0,L) & M^*(0,L) & 1 + V^*(L,L) & -M^*(L,L) \\ -v^{*\prime}(0,L) & M^{*\prime}(0,L) & V^{*\prime}(L,L) & 1 - M^{*\prime}(L,L) \end{bmatrix} \begin{Bmatrix} U_2 \\ U_6 \\ U_8 \\ U_{12} \end{Bmatrix} = \begin{bmatrix} -w^*(0,0) & -S^*(0,0) & -w^*(L,0) & -S^*(L,0) \\ -w^{*\prime}(0,0) & -S^{*\prime}(0,0) & -w^{*\prime}(L,0) & -S^{*\prime}(L,0) \\ -w^*(0,L) & -S^*(0,L) & -w^*(L,L) & -S^*(L,L) \\ -w^{*\prime}(0,L) & -S^{*\prime}(0,L) & -w^{*\prime}(L,L) & -S^{*\prime}(L,L) \end{bmatrix} \begin{Bmatrix} F_2 \\ F_6 \\ F_8 \\ F_{12} \end{Bmatrix} + \begin{Bmatrix} \int_0^L w^*(x, 0)q(x)\,dx \\ \int_0^L w^{*\prime}(x, 0)q(x)\,dx \\ \int_0^L w^*(x, L)q(x)\,dx \\ \int_0^L w^{*\prime}(x, L)q(x)\,dx \end{Bmatrix} \quad (23)$$

To evaluate the foregoing matrices, we need the fundamental solution w^* and its derivatives. For a uniform beam, they are given by Beskos (1989)

$$w^* = -\text{sgn}(x - \xi)\frac{(x - \xi)^3}{12EI_0}; \quad S^* = -\text{sgn}(x - \xi)\frac{(x - \xi)^2}{4EI_0} \quad (24a,b)$$

$$M^* = -\text{sgn}(x - \xi)\frac{(x - \xi)}{2}; \quad V^* = -\frac{1}{2}\text{sgn}(x - \xi) \quad (24c,d)$$

Since the change in stiffness does not effect V^* and M^*, (24c) and (24d) are also valid for nonprismatic beams. Therefore, we can obtain S^* by integrating M^*, i.e.

$$S^*(x, \xi) = \int_\xi^x \frac{M^*}{EI(x)}\,dx \quad (25)$$

Substituting $I(x)$ from (2c)

$$S^*(x, \xi) = \frac{L^3}{4EI_0}\frac{(-x + \xi)^2}{(L + cx)^2(L + c\xi)}, \; n = 1 \quad (26a)$$

$$S^*(x, \xi) = \frac{L^4}{12EI_0}\frac{(-x + \xi)^2(3L + cx + 2c\xi)}{(L + cx)^3(L + c\xi)^2}, \; n > 1 \quad (26b)$$

Integrate S^* to get w^*

$$w^*(x, \xi) = \frac{L^3}{4EI_0c^3}\left[1 + \frac{c(x - \xi)}{L + c\xi} - \frac{L + c\xi}{L + cx} - 2\log\frac{L + cx}{L + c\xi}\right], \; n = 1 \quad (27a)$$

$$w^*(x, \xi) = \frac{L^4}{12EI_0}\frac{(x - \xi)^3}{(L + cx)^2(L + c\xi)^2}, \; n > 1 \quad (27b)$$

Using (26) and (27) in (23), we get

$$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & \frac{L}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{L}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_6 \\ U_8 \\ U_{12} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \beta & \gamma \\ 0 & 0 & \delta & \varepsilon \\ \beta & -\gamma & 0 & 0 \\ -\delta & \varepsilon & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} F_2 \\ F_6 \\ F_8 \\ F_{12} \end{Bmatrix} + \begin{Bmatrix} \int_0^L w^*(x, 0)q(x)\,dx \\ \int_0^L w^{*\prime}(x, 0)q(x)\,dx \\ \int_0^L w^*(x, L)q(x)\,dx \\ \int_0^L w^{*\prime}(x, L)q(x)\,dx \end{Bmatrix}$$

where

$$\beta = \frac{L^3}{EI}\frac{2c + c^2 - 2(1 + c)\log(1 + c)}{4c^3(1 + c)} \quad (29a)$$

$$\gamma = \frac{L^2}{EI}\frac{1}{4(1 + c)^2}; \quad \delta = -\frac{L^2}{EI}\frac{1}{4(1 + c)} \quad (29b,c)$$

$$\varepsilon = -\frac{L}{EI}\frac{2 + c}{4(1 + c)^2}, \ n = 1 \qquad (29d)$$

$$\beta = \frac{L^3}{EI}\frac{1}{12(1 + c)^2}; \quad \gamma = \frac{L^2}{EI}\frac{(3 + c)}{12(1 + c)^3} \qquad (29e,f)$$

$$\delta = -\frac{L^2}{EI}\frac{(3 + 2c)}{12(1 + c)^2} \qquad (29g)$$

$$\varepsilon = -\frac{L}{EI}\frac{(3 + 3c + c^2)}{6(1 + c)^3}, \ n = 2 \qquad (29h)$$

As before, the stiffness equations can be obtained by multiplying (28) by the inverse of the matrix on the its right-hand side to get

$$\begin{bmatrix} K_{2,2} & K_{2,6} & K_{2,8} & K_{2,12} \\ & K_{6,6} & K_{6,8} & K_{6,12} \\ \text{symmetric} & & K_{8,8} & K_{8,12} \\ & & & K_{12,12} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_6 \\ U_8 \\ U_{12} \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_6 \\ F_8 \\ F_{12} \end{Bmatrix} + \begin{Bmatrix} P_2 \\ P_6 \\ P_8 \\ P_{12} \end{Bmatrix} \qquad (30)$$

where

$$K_{2,2} = \frac{EI_0}{L^3}\frac{c^3(2 + c)}{KK}; \quad K_{2,6} = \frac{EI_0}{L^2}\frac{c^3}{KK}; \quad K_{2,8} = -K_{2,2} \qquad (31a\text{–}c)$$

$$K_{2,12} = \frac{EI_0}{L^2}\frac{c^3(1 + c)}{KK} \qquad (d)$$

$$K_{6,6} = \frac{EI_0}{L}\frac{-2c - 3c^2 + 2(1 + 2c + c^2)\log(1 + c)}{KK} \qquad (31e)$$

$$K_{6,8} = -K_{2,6} \qquad (31f)$$

$$K_{6,12} = \frac{EI_0}{L}\frac{(1 + c)(2c + c^2) - (2 + c)\log(1 + c)}{KK} \qquad (31g)$$

$$K_{8,8} = K_{2,2}; \quad K_{8,12} = -K_{2,12} \qquad (31h,i)$$

$$K_{12,12} = \frac{EI_0}{L}\frac{(1 + c)^2(-2c + c^2 + 2)\log(1 + c)}{KK}, \ n = 1 \qquad (31j)$$

where

$$KK = -2c + (2 + c)\log(1 + c) \qquad (31k)$$

and

$$K_{2,2} = \frac{EI_0}{L^3}[4(1 + c)(3 + 3c + c^2)] \qquad (31l)$$

$$K_{2,6} = \frac{EI_0}{L^2}[2(1 + c)(3 + c)] \qquad (31m)$$

$$K_{2,8} = -K_{2,2}; \quad K_{2,12} = \frac{EI_0}{L^2}[2(1 + c)^2(3 + 2c)] \qquad (31n,o)$$

$$K_{6,6} = \frac{EI_0}{L}[4(1 + c)] \qquad (31p)$$

$$K_{6,8} = -K_{2,6}; \quad K_{6,12} = \frac{EI_0}{L}[2(1 + c)^2]; \quad K_{8,8} = K_{2,2} \qquad (31q\text{–}s)$$

$$K_{8,12} = -K_{2,12}; \quad K_{12,12} = \frac{EI_0}{L}[4(1 + c)^3], \ n = 2 \qquad (31t\text{–}u)$$

The last column matrix in (30) is given by

$$\begin{Bmatrix} P_2 \\ P_6 \\ P_8 \\ P_{12} \end{Bmatrix} = \frac{1}{\beta\varepsilon - \gamma\delta}\begin{bmatrix} 0 & 0 & \varepsilon & \gamma \\ 0 & 0 & \delta & \beta \\ \varepsilon & -\gamma & 0 & 0 \\ -\delta & \beta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \int_0^L w^*(x, 0)q(x)\,dx \\ \int_0^L w^{*\prime}(x, 0)q(x)\,dx \\ \int_0^L w^*(x, L)q(x)\,dx \\ \int_0^L w^{*\prime}(x, L)q(x)\,dx \end{Bmatrix} \qquad (32)$$

which, again, depends on the type of loading function $q(x)$ as discussed in the following section.

Similar results for stiffnesses and fixed-end actions can be obtained for the case of flexure in the xz-plane (i.e., $K_{3,3}$, $K_{3,5}$, $\ldots$, $K_{11,11}$, and P_3, P_5, P_9, P_{11}).

Member-End Actions due to Member Loads (Column Matrix P)

To complete the stiffness analysis, the member-end actions, i.e., fixed-end axial forces P_1 and P_7; fixed-end torques P_4 and P_{10}; fixed-end shears P_2 and P_8; and fixed-end moments P_6 and P_{12} must be determined for the given load function. We will consider the two cases of concentrated and uniform loads.

End Actions due to Concentrated Loads

The loading function can be represented by $p_c\Delta(x - x_c)$, $t_c\Delta(x - x_c)$, and $q_c\Delta(x - x_c)$ for the axial, torsional, and flexural cases, respectively, where p_c, t_c, and q_c are the respective magnitudes and $\Delta(x - x_c)$ is the Dirac delta function which represents a unit charge applied at the point $x = x_c$. The fixed-end axial forces, torques, shear forces, and moments can be calculated from (16), (17), and (32). The results are given in Appendix I.

End Actions due to Uniform Load

Let the load be represented by p_u, t_u, and q_u for the axial, torsional, and flexure cases, respectively, and using the same procedure, obtain the fixed-end actions given in Appendix I.

The end actions for other types of loading can be obtained by substituting the loading functions in (16) and (32).

NUMERICAL CHECKS

To check the correctness of the foregoing stiffness expressions, the following examples are considered.

Example 1

Consider a tapered beam with a thin circular section as shown in Fig. 2. The beam is assumed to have a lineraly varying diameter while the thickness is kept constant. According to the geometry, $c = 1$, $n = 1$, and therefore the results of the exact stiffnesses as calculated from (15*a*,*b*), (18*b*,*c*), and (31*a*)–(31*j*) are given in the third column of Table 1. The fourth column of Table 1 contains the corresponding stiffnesses if the beam were represented by 100 uniform segments. Table 1 shows that all the exact stiffnesses agree to about 0.038% or better with the numerical results.

Example 2

Consider a beam similar to that of example 1 but with a solid circular cross section which has a diameter increasing linearly from 100 mm at one end to 200 mm at the other end. In this case: $c = 1$ and $n = 2$, and the results of the exact stiffnesses as calculated from (15*c*,*d*), (18), and (31*l*)–(31*u*) are compared to those obtained for the corresponding stepped beam (with 100 uniform segments) as given in Table 2. The results, once again, are in excellent agreement.

Example 3

As a final example, consider the frame in Fig. 3. The frame is composed of four members and the stiffness of each member is varied in a way such that $n = 1$. The axial deformation were

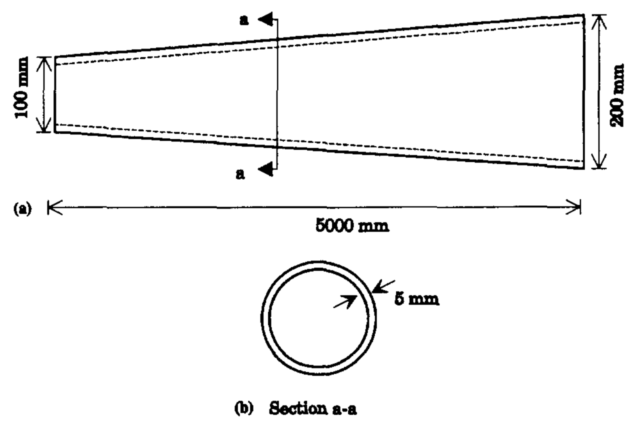

FIG. 2. Geometry of Tapered Beam (Example 1)

TABLE 1. Stiffness Components for $n = 1$

Action (1)	Stiffness (2)	Exact (3)	Numerical ($N = 100$) (4)	Percent difference (5)
Axial	$K_{1,1}$	119.3823000	119.3805000	−0.0014507
	$K_{1,7}$	−119.3823000	−119.3805000	0.0014507
	$K_{7,7}$	119.3823000	119.3805000	−0.0014507
Torsional	$K_{4,4}$	0.1381337	0.1381294	−0.0031500
	$K_{4,10}$	−0.1381337	−0.1381294	0.0031500
	$K_{10,10}$	0.1381337	0.1381294	−0.0031500
Flexural (XY-plane)	$K_{2,2}$	0.1508697	0.1508373	−0.0214571
	$K_{2,6}$	0.2155514	0.2154818	−0.0322805
	$K_{2,8}$	−0.1508697	−0.1508373	−0.0214571
	$K_{2,12}$	0.5387976	0.5387046	−0.0172716
	$K_{6,6}$	0.5388499	0.5387046	−0.0269862
	$K_{6,8}$	−0.2155514	−0.2154818	−0.0322805
	$K_{6,12}$	0.5389085	0.5387046	−0.0378625
	$K_{8,8}$	0.1508697	0.1508373	−0.0214571
	$K_{8,12}$	−0.5387976	−0.5387046	−0.0172716
	$K_{12,12}$	2.1550800	2.1548180	−0.0121266

TABLE 2. Stiffness Components for $n = 2$

Action (1)	Stiffness (2)	Exact (3)	Numerical ($N = 100$) (4)	Percent difference (5)
Axial	$K_{1,1}$	86.1152300	86.1148500	−4.4297790E-004
	$K_{1,7}$	−86.1152300	−86.1148500	4.4297790E-004
	$K_{7,7}$	86.1152300	86.1148500	−4.4297790E-004
Torsional	$K_{4,4}$	0.1776048	0.1775949	−0.0055461
	$K_{4,10}$	−0.1776048	−0.1775949	0.0055461
	$K_{10,10}$	0.1776048	0.1775949	−0.0055461
Flexural (XY-plane)	$K_{2,2}$	0.1017345	0.1017172	−0.0170009
	$K_{2,6}$	0.1695687	0.1695286	−0.0236357
	$K_{2,8}$	−0.1017345	−0.1017172	−0.0170009
	$K_{2,12}$	0.3391006	0.3390572	−0.0127803
	$K_{6,6}$	0.4622070	0.4621159	−0.0197085
	$K_{6,8}$	−0.1695687	−0.1695286	−0.0236357
	$K_{6,12}$	0.3856321	0.3855272	−0.0272106
	$K_{8,8}$	0.1017345	0.1017172	−0.0170009
	$K_{8,12}$	0.3391006	−0.3390572	−0.0127803
	$K_{12,12}$	1.3098710	1.3097590	−0.0085282

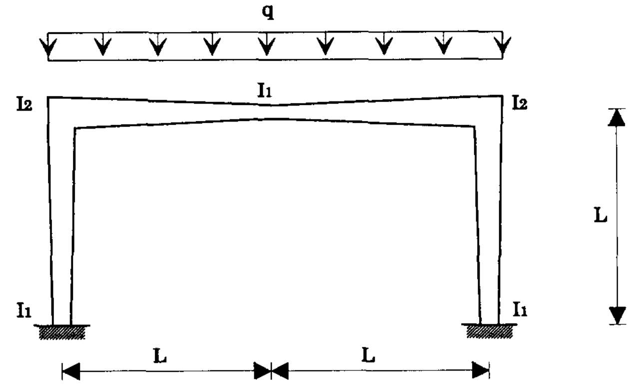

FIG. 3. Frame with Variable Cross-Section Members (Example 3)

TABLE 3. Results for Frame with Nonprismatic Members

Number of segments per member (1)	δ (2)	θ (3)
10	−0.02347	−0.01830
20	−0.02369	−0.01845
30	−0.02372	−0.01848
4 (Exact)	−0.02375	−0.01850

restrained in all members. The unitless data used in the analysis is: $q = 1$, $L = 1$, $I_1 = 1$, $I_2 = 2$, and $E = 1$. The solution based on the derived stiffness expressions is compared to that obtained using the structural analysis package "STRUDL" by dividing each member into 10, 20, and 30 uniform segments as given in Table 3. In Table 3, δ is the vertical deflection along the centerline, and θ is the rotation at the left corner. The results show the convergence of the numerical solution to the one based on exact stiffnesses. However, the data preparation time as well as the computer time are cut drastically using the present analysis.

CONCLUSIONS

In this paper, a simple, yet exact, method of deriving closed-form expressions for the axial, torsional, flexural stiffnesses, and fixed-end actions for a class of tapered members was presented. It is well known that the only source of approximation in the boundary integral method is the approximate modeling of the boundary. The boundary in the present case is simply the two ends of the member over which the boundary conditions are satisfied exactly. The correctness of the derived expressions is verified using numerical examples. The derived expressions are useful for problems of shape optimization of tapered members. They also can be combined with any space frame program to perform exact analysis of frames consisting of tapered members.

APPENDIX I. FIXED-END ACTIONS

End Actions for Concentrated Loads

Define $\bar{x} = x_c L$.

Fixed-end axial forces

$$P_1 = p_c\left[1 - \frac{\log(1 + c\bar{x})}{\log(1 + c)}\right]; \quad P_7 = p_c - P_1,\ n = 1 \qquad (33a,b)$$

$$P_1 = p_c \frac{1 - \bar{x}}{1 + c\bar{x}}; \quad P_7 = p_c - P_1,\ n = 2 \qquad (33c,d)$$

Fixed-end torques

$$P_4 = t_c \frac{(1 - \bar{x})(2 + c + c\bar{x})}{(2 + c)(1 + c\bar{x})^2}; \quad P_{10} = t_c - P_4, \; n = 1 \qquad (34a,b)$$

$$P_4 = \frac{t_c}{c(3 + 3c + c^2)} \left[\left(\frac{1 + c}{1 + c\bar{x}} \right)^3 - 1 \right]; \quad P_{10} = t_c - P_4, \; n = 2 \qquad (34c,d)$$

Fixed-end shears and moments

$$P_2 = q_c \frac{c(2 - 2\bar{x} + c\bar{x} - c\bar{x}^2) + (2 + c + 2c\bar{x} + c^2\bar{x})\log\left(\frac{1 + c\bar{x}}{1 + c}\right)}{KK(1 + c\bar{x})} \qquad (35a)$$

$$P_6 = q_c L \frac{c\bar{x}(1 - \bar{x}) + \log(1 + c\bar{x}) - \bar{x}(2 - \bar{x})\log(1 + c) + c\bar{x}\log\left(\frac{1 + c\bar{x}}{1 + c}\right)}{KK} \qquad (35b)$$

$$P_8 = q_c - P_2; \quad P_{12} = -q_c x_c - P_6 + P_8 L, \; n = 1 \qquad (35c,d)$$

$$P_2 = q_c \frac{(-1 + \bar{x})^2(1 + 2\bar{x} + 2c\bar{x})}{(1 + c\bar{x})^2} \qquad (35e)$$

$$P_6 = q_c \frac{\bar{x}(1 - \bar{x})^2}{(1 + c\bar{x})^2} \qquad (35f)$$

$$P_8 = q_c - P_2; \quad P_{12} = -q_c x_c - P_6 + P_8 L, \; n = 2 \qquad (35g,h)$$

End Actions for Uniform Load

Fixed-end axial forces

$$P_1 = \frac{p_u L}{c} \left[\frac{1}{\log(1 + c)} - \frac{1}{c} \right]; \quad P_7 = p_u - P_1, \; n = 1 \qquad (36a,b)$$

$$P_1 = p_u L \frac{(1 + c)\log(1 + c) - c}{c^2}; \quad P_7 = p_u - P_1, \; n = 2 \qquad (36c,d)$$

Fixed-end torques

$$P_4 = \frac{t_u L}{2 + c}; \quad P_{10} = t_u - P_4, \; n = 1 \qquad (37a,b)$$

$$P_4 = t_u L \frac{3 + c}{2(3 + 3c + c^2)}; \quad P_{10} = t_u - P_4, \; n = 2 \qquad (37c,d)$$

Fixed-end shears and moments

$$P_2 = q_u L \frac{-6c - c^2 + (6 + 4c)\log(1 + c)}{2cKK} \qquad (38a)$$

$$P_6 = -q_u L^2 \frac{(2c^2 - c^3) - (4c + 3c^2)\log(1 + c) + (2 + 4c + 2c^2)\log^2(1 + c)}{2c^3 KK} \qquad (38b)$$

$$P_8 = q_u L - P_2; \quad P_{12} = -\frac{q_u L^2}{2} - P_6 + P_8 L, \; n = 1 \qquad (38c,d)$$

$$P_2 = q_u L \frac{6c + 9c^2 + 4c^3 - 2(3 + 6c + 4c^2 + c^3)\log(1 + c)}{c^4} \qquad (38e)$$

$$P_6 = q_u L^2 \frac{6c + 5c^2 - 2(3 + 4c + c^2)\log(1 + c)}{2c^4} \qquad (38f)$$

$$P_8 = q_u L - P_2; \quad P_{12} = -\frac{q_u L^2}{2} - P_6 + P_8 L, \; n = 2 \qquad (38g,h)$$

APPENDIX II. REFERENCES

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APPENDIX III. NOTATION

The following symbols are used in this paper:

- $A(x), A_0$ = area of the member cross section;
- c = taper ratio;
- E = elastic modulus;
- $\mathbf{F}$ = force vector;
- G = shear modulus;
- $I(x), I_0$ = moment of inertia of the member cross section;
- $J(x), J_0$ = polar moment of inertia of the member cross section;
- $\mathbf{K}$ = stiffness matrix;
- KK = constant defined by Eq. (31b);
- L = length of member;
- $M(x), M^*$ = bending moment;
- n = a real number which determines taper profile;
- $\mathbf{P}$ = fixed-end actions vector;
- $p(x)$ = distributed axial load;
- $q(x)$ = distributed flexural load;
- $S(x), S^*$ = slope;
- $t(x)$ = distributed torque;
- $\mathbf{U}$ = displacement vector;
- u, u^* = axial displacement;
- V, V^* = shear force;
- w, w^* = flexural displacement;
- xyz = coordinate system;
- α = constant defined by Eq. (13);
- $\beta, \gamma, \delta, \varepsilon$ = constants defined by Eq. (29);
- θ = torsional rotation; and
- ξ = any point along the member.