EXACT ANALYSIS OF NONPRISMATIC BEAMS

By H. J. Al-Gahtani[1] and M. S. Khan[2]

ABSTRACT: An exact analysis of nonprismatic beams with general boundary conditions is presented. The analysis is based on the boundary integral method. The fundamental solutions for nonprismatic beams of linear and parabolic profiles are derived. The fundamental solution is used as a weighing function to transform the governing equation and the boundary condition equations into four algebraic equations. The resulting algebraic equations offer exact relations among the physical variables at the beam supports. The method is tested through two numerical examples to show its accuracy.

INTRODUCTION

Members of variable stiffness are commonly used to optimize the distribution of weight and strength, achieve a better distribution of the internal stresses, reduce the dead load, and sometimes to satisfy architectural and functional requirements in many engineering structures—such as highway bridges, buildings, space and aircraft structures—as well as in many mechanical components and machines. Therefore, the analysis of nonprismatic beams is of interest to many mechanical, aeronautical, and structural engineers. It is a standard engineering practice to analyze beams of uniform and variable depth on the basis of Bernoulli-Euler beam theory, which is quite adequate as long as the tapering ratio is not too great (Boley 1963). Although the solution of the governing equation—which is a fourth-order ordinary differential equation with variable coefficients—can be obtained through successive integration, the procedure becomes tedious when dealing with general depth variation, general loading, and general boundary conditions. Generally, except for some particular cases [e.g., Timoshenko and Young (1965); Hibbeler (1990); Lee et al. (1990)] no closed-form solution is available. The available analytical solutions involve lengthy and tedious calculations. To simplify the method, tables and charts for the stiffness and beam constants are prepared and published by the Portland Cement Association (PCA 1958). However, though these tables simplify the solution, they have the following limitations:

1. They can give the beam constants only for straight and parabolic haunches.
2. They are applicable for two types of loading, i.e., concentrated and uniformly distributed loading. Partial loading cases are not considered.

Thus, a wide variety of approximate and numerical techniques have been developed through the years. In the finite element and finite difference methods, each tapered span is broken into a number of uniform elements (stepped representation) with known uniform stiffness that are superimposed to produce the stiffness of the member. This clearly indicates that the number of equations to be solved increases as the number of elements/spans increases. A detailed literature survey on this topic has been done by Khan (1995).

The objective of this note is to present a simple yet exact procedure for the analysis of nonprismatic beams. The proposed procedure is based on the boundary integral method (Banerjee and Butterfield 1981) and does not require the discretization of the beam span as done in the domain-type methods. The variation of the beam depth is assumed to be general with the requirement of being second-order differentiable. The boundary conditions are also assumed to be general. The proposed procedure is explained through the two cases of linear and parabolic depth variation. The procedure is tested through two numerical examples.

BOUNDARY INTEGRAL EQUATIONS

The governing differential equation for transverse deflection $w(x)$ of a nonprismatic Bernoulli-Euler beam of finite length L, subjected to transverse loads and with general, linear elastically end-restrained boundary conditions as shown in Fig. 1, is

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right) + q(x) = 0, \quad 0 < x < L \tag{1}$$

where $EI(x)$ is variable flexural rigidity and $q(x)$ is the distributed load.

The boundary integral method (BIM) begins by considering the domain of the problem in its entirety, without subdivisions. All the discretization is performed on the boundary. In this case there are only two "nodes" involved, regardless of the complexity of the problem. The two nodes are placed, respectively, at $x = 0$ and at $x = L$ as shown in Fig. 1. The slope $\theta(x)$, moment $M(x)$, and shear $V(x)$ are related to the primary variable, deflection $w(x)$ by

$$\theta(x)=\frac{dw}{dx};\quad M(x)=EI(x)\frac{d^2w}{dx^2};\quad V(x)=\frac{d}{dx}\left(EI(x)\frac{d^2w}{dx^2}\right) \tag{2a-c}$$

The associated boundary conditions at $x = 0$ are

$$V(0) = K_t w(0);\quad M(0) = K_r\theta(0) \tag{3a,b}$$

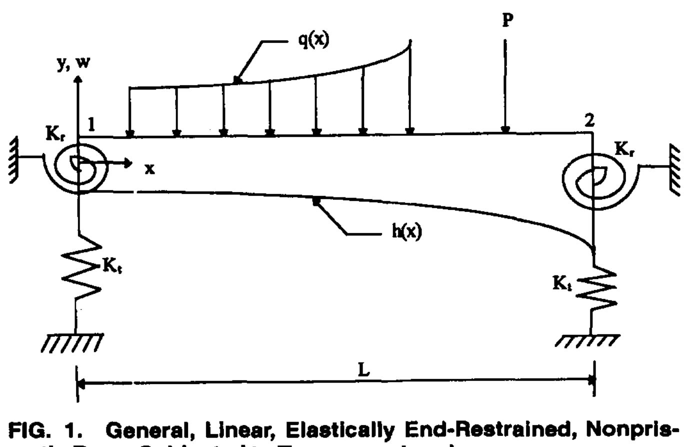

FIG. 1. General, Linear, Elastically End-Restrained, Nonprismatic Beam Subjected to Transverse Loads

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[1]Assoc. Prof., Dept. of Civ. Engrg., King Fahd Univ. of Petroleum and Minerals, Dhahran-31261, Saudi Arabia.
[2]Grad. Student, Dept. of Civ. Engrg., King Fahd Univ. of Petroleum and Minerals, Dhahran-31261, Saudi Arabia.

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Similarly at $x = L$

$$V(L) = K_t w(L); \quad M(L) = K_r \theta(L) \tag{3c,d}$$

where K_t = translational spring constant and K_r = rotational spring constant and therefore the above boundary conditions can represent any of the following cases:

- Free boundary when $K_t = K_r = 0$
- Simple support when $K_t = \infty$, $K_r = 0$
- Fixed support when $K_r = \infty$, $K_t = \infty$
- General elastic support when $0 < K_t < \infty$ and $0 < K_r < \infty$

In order to derive the boundary integral equations, let us follow the procedure given by Butterfield (1979), i.e., multiply the left hand side of (1) by a weighting function w^* and integrate over the domain of the beam (see Banerjee and Butterfield 1981).

$$\int_0^L \left[\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right) + q(x)\right] w^*\, dx = 0 \tag{4}$$

Integrating the first term in (4) by parts, four times, we get

$$\int_0^L w\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w^*}{dx^2}\right) dx + \int_0^L q(x)w^*\, dx$$
$$- [-Vw^* + M\theta^* - \theta M^* + wV^*]_{x=0}^{x=L} = 0 \tag{5}$$

where $\theta^*(x)$, $M^*(x)$, and $V^*(x)$ are related to w^* through the same differential operators given in (2). Let us choose $w^*(x, \xi)$ such that it satisfies the following differential equation:

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w^*}{dx^2}\right) + \delta(x - \xi) = 0 \tag{6}$$

where δ = Dirac delta function, which represents a unit concentrated force at a distance ξ from the origin and has the following property:

$$\int_0^L g(x)\delta(x - \xi)\, dx = g(\xi) \tag{7}$$

Then (4) becomes

$$w(\xi) = [Vw^* - M\theta^* + \theta M^* - wV^*]_{x=0}^{x=L} + \int_0^L q(x)w^*\, dx,$$
$$0 < \xi < L \tag{8}$$

Thus, once the fundamental solution w^* is known, (8) gives the deflection w at any point ξ inside the domain of the beam in terms of the values of deflection, slope, moment, shear at the beam ends, and the given load $q(x)$. Furthermore, (8) may be differentiated with respect to ξ to produce the slope, i.e.

$$\theta(\xi) = \left[V\frac{dw^*}{d\xi} - M\frac{d\theta^*}{d\xi} + \theta\frac{dM^*}{d\xi} - w\frac{dV^*}{d\xi}\right]_{x=0}^{x=L}$$
$$+ \int_0^L q(x)\frac{dw^*}{d\xi}\, dx; \quad 0 < \xi < L \tag{9}$$

The BIM proceeds from this point by first solving for the unknown boundary data (that has not been specified) in terms of that which has been specified. To successfully implement this procedure, four simultaneous equations are obtained in the following matrix form by applying (8) and (9) at the boundary points, i.e., $\xi \to 0$ and $\xi \to L$ (Banerjee and Butterfield 1981):

$$\begin{bmatrix} 1 - V^*(0,0) & M^*(0,0) & V^*(L,0) & -M^*(L,0) \\ -V^{*\prime}(0,0) & 1 + M^{*\prime}(0,0) & V^{*\prime}(L,0) & -M^{*\prime}(L,0) \\ -V^*(0,L) & M^*(0,L) & 1 + V^*(L,L) & -M^*(L,L) \\ -V^{*\prime}(0,L) & M^{*\prime}(0,L) & V^{*\prime}(L,L) & 1 - M^{*\prime}(L,L) \end{bmatrix}$$
$$\cdot \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} -w^*(0,0) & \theta^*(0,0) & w^*(L,0) & -\theta^*(L,0) \\ -w^{*\prime}(0,0) & \theta^{*\prime}(0,0) & w^{*\prime}(L,0) & -\theta^{*\prime}(L,0) \\ -w^*(0,L) & \theta^*(0,L) & w^*(L,L) & -\theta^*(L,L) \\ -w^{*\prime}(0,L) & \theta^{*\prime}(0,L) & w^{*\prime}(L,L) & -\theta^{*\prime}(L,L) \end{bmatrix}$$
$$\cdot \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} + \begin{Bmatrix} \int_0^L q(x)w^*(x,0)\, dx \\ \int_0^L q(x)w^{*\prime}(x,0)\, dx \\ \int_0^L q(x)w^*(x,L)\, dx \\ \int_0^L q(x)w^{*\prime}(x,L)\, dx \end{Bmatrix} \tag{10}$$

where the prime indicates differentiation with respect to ξ, and the subscripts 1 and 2 indicate the node at $x = 0$ and $x = L$, respectively. The above equation can be written in short

$$[\mathbf{H}]\{\mathbf{\Delta}\} = [\mathbf{G}]\{\mathbf{F}\} + \{\mathbf{B}\}$$

where the elements of the matrices **H**, **G**, and **B** are the limiting values of the fundamental solution and its derivatives at the two ends of the beam. Since unknown values of $\{\mathbf{\Delta}\}$ align with known values of $\{\mathbf{F}\}$, the system of equations may be rearranged into the following form:

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{Z}\} \tag{11}$$

where $\{\mathbf{X}\}$ contains all the boundary unknowns and $\{\mathbf{Z}\}$ contains the algebraic sum of the products of all known boundary values with their corresponding columns of $[\mathbf{G}]$ or $[\mathbf{H}]$ as the case may be.

Once the system (11) is solved, all the boundary values will be known. These are substituted into the domain equation for deflection, i.e., (8), and the resulting equation can be evaluated at any point ξ. The same is true for the domain equation for slope, i.e., (9). Also (9) may be differentiated further to obtain the shearing force and the bending moment at any point ξ.

DERIVATION OF FUNDAMENTAL SOLUTION

Linear Variation

The fundamental solution, w^*, of the differential equation governing the beams of variable depth can be derived as follows: For instance, if the depth of a rectangular beam varies linearly

$$h(x) = h_0\left(1 + \frac{bx}{L}\right)$$

where h_0 = depth at the origin; and b = constant such that the beam becomes prismatic when $b = 0$. Assuming that the width of the beam is constant, (6) becomes

$$\frac{d^2}{dx^2}\left[EI_0\left(1 + \frac{bx}{L}\right)^3 \frac{d^2w}{dx^2}\right] + \delta(x - \xi) = 0 \tag{12}$$

where I_0 = moment of inertia at $x = 0$. The integration of (12) yields

$$\frac{d}{dx}\left[EI_0\left(1 + \frac{bx}{L}\right)^3 \frac{d^2w^*}{dx^2}\right] = V^* = \begin{cases} C_1 & x < \xi \\ -(1 - C_1) & x > \xi, \end{cases}$$

where $0 < C_1 < 1$

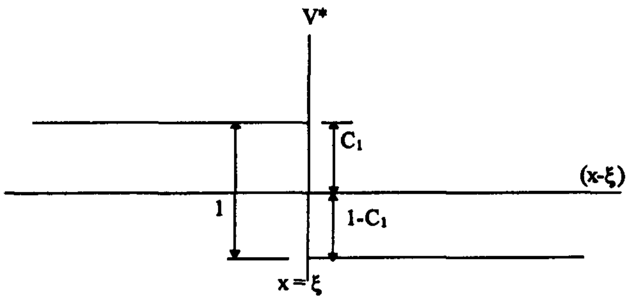

FIG. 2. Function V^*

where C_1 is a constant. The function V^* is plotted as shown in Fig. 2, and it is clear from the figure that the slope of V^* is zero everywhere except at $x = \xi$ where it becomes $-\infty$, as it should according to (12). For symmetry let us choose $C_1 = 1/2$. Therefore

$$V^* = -\frac{1}{2}\,\mathrm{sgn}(x - \xi) \tag{13a}$$

where

$$\mathrm{sgn}(x - \xi) = \begin{cases} -1 & x < \xi \\ 1 & x > \xi \end{cases}$$

Integrate V^* to get M^*:

$$M^* = -\mathrm{sgn}(x - \xi)\,\frac{(x - \xi)}{2} \tag{13b}$$

Integrate M^* with respect to x, between ξ and x to get θ^*:

$$\theta^* = \int_\xi^x \frac{M^*\,dx}{EI_0\left(1 + \dfrac{bx}{L}\right)^3} = -\mathrm{sgn}(x - \xi)\,\frac{L^3(x - \xi)^2}{4EI_0(L + bx)^2(L + b\xi)} \tag{13c}$$

Similarly, integrate θ^* with respect to x, between ξ and x to get w^*:

$$w^* = -\mathrm{sgn}(x - \xi)\,\frac{L^3}{4b^2EI_0}\left[(x - \xi)\left\{\frac{1}{L + bx} + \frac{1}{L + b\xi}\right\} - \frac{2}{b}\log\frac{L + bx}{L + b\xi}\right] \tag{13d}$$

It should be noted that w^* represents the deflection at point x due to a unit concentrated force applied at ξ.

Parabolic Variation

If the depth of beam varies parabolically, i.e.

$$h(x) = h_0\left(1 + \frac{bx^2}{L^2}\right)$$

and the governing equation becomes

$$\frac{d^2}{dx^2}\left[EI_0\left(1 + \frac{bx^2}{L^2}\right)^3\frac{d^2w}{dx^2}\right] + f(x) = 0, \quad 0 < x < L \tag{14}$$

Following the same procedure, we get

$$\theta^* = -\mathrm{sgn}(x - \xi)\,\frac{L^2}{16bEI_0}\left[\frac{-2L^2(L^2 + bx\xi)}{(L^2 + bx^2)^2} - \frac{3bx\xi}{(L^2 + bx^2)} + \frac{2L^2 + 3b\xi^2}{(L^2 + b\xi^2)} - 3\sqrt{b}L\xi(\gamma_1 - \gamma_2)\right] \tag{15a}$$

where $\gamma_1 = \tan^{-1}(x\sqrt{b})$ and $\gamma_2 = \tan^{-1}(\xi\sqrt{b})$. And

$$w^* = -\mathrm{sgn}(x - \xi)\,\frac{1}{16bEI_0}\left[\frac{(\xi - x)}{(L^2 + bx^2)} - \frac{2L^4(\xi - x)}{(L^2 + b\xi^2)} - \frac{(L^3 + 3bLx\xi)(\gamma_1 - \gamma_2)}{\sqrt{b}} + \frac{3bL^2(x\xi^2 - \xi^3)}{(L^2 + b\xi^2)}\right] \tag{15b}$$

Similarly, fundamental solutions can be derived for other types of stiffness variations.

NUMERICAL EXAMPLES

Example 1

Consider the following problem of a single span nonprismatic beam AB of unit length as shown in Fig. 3. The depth of the beam varies parabolically from depth $h = 1.0$ units at the left end to $h = 2.0$ units at the right end (i.e., $b = 1$) and has a unit width. The beam is supported on a translational spring, with the spring stiffness $K_t = 10$ at the left end, fixed at the right end and carrying uniformly distributed load, $q = 1.0$.

This problem has been solved using finite-element-method (FEM)-based STRUDL (1977) software, by dividing the beam into 10 and 20 prismatic elements where I is based on the center of each element. STRUDL results for bending moment, deflection, rotation, and support reactions are given in Table 1, where the first column represents the results obtained by BIM and the last two columns show the results obtained by STRUDL.

Example 2

Consider the following problem of a three-span bridge girder of variable depth as shown in Fig. 4. The flexural rigidity of the girder EI varies parabolically from depth $h = 2.5$ units at both ends and midspan to $h = 7.5$ units at the intermediate supports. In continuous beams, the procedure is basically the same as for single span beams. The extra thing we have to keep in mind is the connectivity between the spans. The continuity and equilibrium conditions at the intermediate support are

$$M_b = -M_a \quad \text{and} \quad \theta_b = \theta_a$$

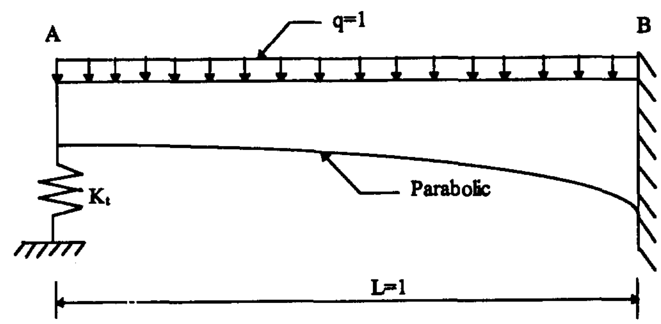

FIG. 3. Beam with Parabolically Varying Depth, Elastically Restrained at One End and Fixed at Other End

TABLE 1. Deflection, Rotation, Bending Moment, and Support Reactions

	BIM	STRUDL	
Variable (1)	1 Nonprismatic region (2)	10 Prismatic elements (3)	20 Prismatic elements (4)
w_A	0.1579	0.1435	0.1579
R_A	1.5792	1.435	1.5792
θ_A	1.2643	1.1375	1.2622
M_B	4.6067	4.133	4.5346
R_B	3.4214	3.0781	3.4033

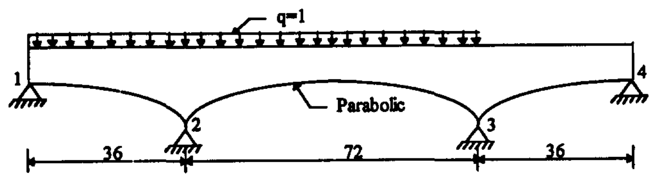

FIG. 4. Continuous Bridge Girder with Parabolically Varying Depth

TABLE 2. Bending Moments, Slopes, and Support Reactions

Variable (1)	Slope-Deflection (2)	BIM: 3 Nonprismatic regions (3)	STRUDL: 20 Prismatic elements[a] (4)	STRUDL: 40 Prismatic elements[a] (5)	STRUDL: 80 Prismatic elements[a] (6)
M_2	594.0	593.75	589.13	592.56	593.44
M_3	453.0	452.81	450.39	452.18	452.64
θ_1	289.20	290.25	288.56	289.20	290.0
θ_2	423.84	423.92	426.12	425.58	424.66
θ_3	614.89	615.50	618.35	617.85	616.39
θ_4	772.76	775.54	776.80	776.24	775.51
R_1	1.50	1.51	1.64	1.54	1.52
R_2	72.85	72.45	72.29	72.41	72.44
R_3	46.15	46.62	46.58	46.61	46.62
R_4	−12.5	−12.58	−12.5	−12.6	−12.57

[a]Total number of elements for 3 spans.

where M_b, θ_b are the moment and rotation before the support, and M_a, θ_a are the moment and rotation after the support.

An analytical solution using slope deflection and Portland Cement Association tables for this problem is available in Timoshenko and Young (1965). This problem has also been solved using FEM-based software STRUDL (1977) by dividing the beam into 20, 40, and 80 elements, and the results of bending moments, slopes, and support reactions are given in Table 2. In that table, the first column represents the analytical results, the second shows the results obtained by BIM, and the last three show the results obtained by STRUDL.

CONCLUSIONS

In this paper, a boundary integral solution of continuous nonprismatic beams has been presented. The method involves no approximation. It is well known that the only source of approximation in the boundary integral method is the approximate modeling of the boundary. The boundary in the present case is simply the ends of each span (beam supports) over which the boundary conditions are satisfied exactly. In addition to the accuracy, the method has more advantages over other numerical methods in terms of the number of equations involved in the solution. Although the method presented is for rectangular beams with linear and parabolic depth variations, it can be applied easily to other cases by modifying the expression for I and following the same procedure.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- b = variation in depth of beam;
- C_1 = arbitrary constant;
- $EI(x)$ = flexural rigidity;
- $h(x)$ = depth of beam;
- h_0 = depth of beam at origin of coordinate system;
- I_0 = moment of inertia at origin of coordinate system;
- K_r = rotational spring constant;
- K_t = translational spring constant;
- L = length of beam;
- $M(x)$ = bending moment;
- M_i = bending moment at support i;
- $q(x)$ = distributed load;
- R_i = reaction at support i;
- $V(x)$ = shear force;
- $w(x)$ = transverse deflection;
- w_i = deflection at support i;
- w^* = weighting function;
- δ = Dirac delta function;
- $\theta(x)$ = slope of beam;
- θ_i = slope at support i; and
- ξ = any point in beam.