A third interpretation of Eq. (4.47) is provided by the groundwater (seepage) flow problem. In this case, u denotes the piezometric head (measured from the bottom of an aquifer) and f is the charge (pumping is negative). The flow velocities are given by

$$v_1 = -a_{11}\frac{\partial u}{\partial x} \qquad v_2 = -a_{22}\frac{\partial u}{\partial y} \tag{4.71}$$

where a_{11} and a_{22} are the coefficients of permeability. (In the present example, we have $a_{11} = a_{22} = 1$.)

The last example of the problem considered here is one of a 2-inch-square (elastic) membrane fixed on the boundary and subjected to uniformly distributed load of unit intensity. In this case, u denotes the transverse deflection of the membrane. ■

The next two examples are concerned with the irrotational flow of an ideal fluid (i.e., a nonviscous fluid). Examples of physical problems that can be approximated by such flows are provided by flow around bodies such as weirs, airfoils, buildings, and so on, and by flow of water through the earth and dams. The equation governing these flows is a special case of Eq. (4.1), namely, the Laplace equation. Therefore, one can use the finite-element equations developed earlier to model these physical problems. Due to the nonrectangular boundaries involved in the two examples to be discussed, only triangular-element meshes were employed. It is possible to use meshes consisting of both triangular and rectangular elements to obtain the same discretization accuracy. However, in the interest of brevity, we will not use meshes with two different kinds of elements. We will return to these examples in Sec. 4.8 on the computer implementation of the finite-element method.

Example 4.2 (Confined flow about a circular cylinder) The irrotational flow of an ideal fluid (i.e., a nonviscous fluid) about a circular cylinder (placed with its axis perpendicular to the plane of the flow between two long horizontal walls; see Fig. 4.11a) is to be analyzed using the finite-element method. The equation governing the flow is given by

$$-\nabla^2 u = 0 \quad \text{in } \Omega \tag{4.72}$$

where u can be one of two functions: (1) u is the *stream function* or (2) u is the *velocity potential*. If u is the stream function ψ, the velocity components of the flow field are given by

$$u_1 = \frac{\partial \psi}{\partial y} \qquad u_2 = -\frac{\partial \psi}{\partial x} \tag{4.73}$$

If u is the velocity potential ϕ, the velocity components can be computed from

$$u_1 = \frac{\partial \phi}{\partial x} \qquad u_2 = \frac{\partial \phi}{\partial y} \tag{4.74}$$

In either case, the velocity field is not affected by a constant term in the solution u.

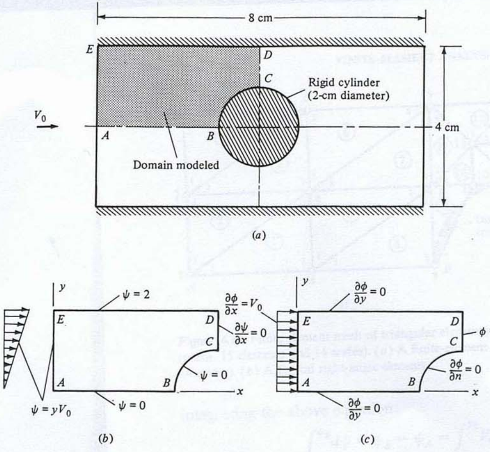

Figure 4.11 Domain and boundary conditions for the stream function and velocity potential formulations of irrotational flow about a cylinder. (*a*) Domain of the flow about a circular cylinder. (*b*) Stream function formulation. (*c*) Velocity potential formulation.

Stream function formulation The boundary conditions on the stream function ψ can be determined as follows (see Fig. 4.11*b*). Streamlines have the property that flow perpendicular to a streamline is zero. Therefore, the fixed walls correspond to streamlines. Due to the biaxial symmetry about the horizontal and vertical centerlines, only a quadrant (say, *ABCDE*) of the domain need be used in the analysis. The fact that the velocity component perpendicular to the horizontal line of symmetry is equal to zero allows us to use that line as a streamline. Since the velocity field depends on the relative difference of two streamlines, we take the value of the streamline that coincides with the axis of symmetry of the cylinder to be zero ($\psi_A = 0$) and then determine the value of ψ on the upper wall from the condition

$$\frac{\partial \psi}{\partial y} = V_0$$

where V_0 is the velocity of the fluid parallel to the streamline. Since V_0 is given only at the inlet, we determine the value of the streamline at point E by