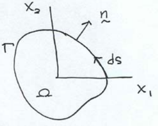
BEM for $a_0\nabla^2 u = 0$

$$a_0 \int_\Omega (\nabla^2 u) w \, d\Omega = 0$$



$$a_0 \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) w \, d\Omega = 0$$

$$a_0 \int_\Omega \left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1} w\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2} w\right)\right] d\Omega - a_0 \int_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_2}\right) d\Omega = 0$$

$$a_0 \int_\Gamma w \left(\frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2\right) ds - a_0 \int_\Omega \left[\frac{\partial}{\partial x_1}\left(u \frac{\partial w}{\partial x_1}\right) + \cancel{u\frac{\partial w}{\partial x_2}} \frac{\partial}{\partial x_2}\left(u \frac{\partial w}{\partial x_2}\right)\right] d\Omega -$$

$$+ a_0 \int_\Omega \left[u \left(\frac{\partial^2 w}{\partial x_1^2}\right) + u \frac{\partial^2 w}{\partial x_2^2}\right] d\Omega = 0$$

$$\int_\Gamma w \, a_0 \left(\frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2\right) ds - \int_\Gamma u \, a_0 \left(\frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2\right) ds$$

$$+ \int_\Omega u \, (a_0 \nabla^2 w) \, d\Omega = 0 \quad \text{—(1)}$$

Let us select w such that

$$\nabla^2 w + \Delta^i = 0 \quad \text{where} \quad \Delta^i = \begin{cases} 0 & \underset{\sim}{x} \neq \underset{\sim}{x}^i \\ \infty & \underset{\sim}{x} = \underset{\sim}{x}^i \end{cases}$$

and has the following properties:

$$\int_\infty \Delta^i \, d\Omega = 1 \quad , \quad \int_\infty f(\underset{\sim}{x}) \Delta^i \, d\Omega = f(\underset{\sim}{x}^i)$$

Then (1) becomes:

$$u(\underset{\sim}{x}^i) = \int_\Gamma w \, q \, ds - \int_\Gamma u \, Q \, ds$$

In the computer code, I'm using u^* and q^* for w and Q, respectively. So we have

$$u(\underset{\sim}{x}^i) = \int_\Gamma u^* q \, ds - \int_\Gamma u q^* \, ds$$

where $u^* \cancel{(}\underset{\sim}{x}, \underset{\sim}{x}^i) = \frac{1}{2\pi a_0} \ln \frac{1}{r}$

$$q^*(\underset{\sim}{x}, \underset{\sim}{x}^i) = \frac{\partial u^*}{\partial n} = \overbrace{\frac{\partial u^*}{\partial x_1}}^{q_1^*} n_1 + \overbrace{\frac{\partial u^*}{\partial x_2}}^{q_2^*} n_2$$

and $r = \sqrt{(x_1 - x_1^i)^2 + (x_2 - x_2^i)^2}$

So that $q_1^* = \frac{\cancel{2}(x_1 - x_1^i)}{\cancel{2}\pi a_0 r^2} = -\frac{(x_1 - x_1^i)}{r^2}$

$$q_2^* = -\frac{(x_2 - x_2^i)}{r^2}$$

$$q^* = -\frac{(x_1 - x_1^i) n_1 + (x_2 - x_2^i) n_2}{r^2}$$

The final eq. becomes:

$$\cancel{u(\underset{\sim}{x}^i) = \int_\Gamma u}$$

$$u(\underset{\sim}{x}^i) = \frac{1}{2\pi a_0} \int_\Gamma q \left(\ln \frac{1}{r}\right) ds + \int_\Gamma u \left[\frac{(x_1 - x_1^i) n_1 + (x_2 - x_2^i) n_2}{r^2}\right] ds$$

Derivation of domain integral eqs using indicial notation

$$\int_\Omega (u_{,ii})\, w \, d\Omega = 0$$

$$\int_\Omega \left[(u_{,i}\, w)_{,i} - u_{,i}\, w_{,i}\right] d\Omega = 0$$

$\Downarrow$ Div. theorem

$$\int_\Gamma w\, u_{,i}\, n_i \, ds - \int_\Omega u_{,i}\, w_{,i}\, d\Omega = 0$$

$$\int_\Gamma w\, u_{,i}\, n_i\, ds - \left[\int_\Omega (u\, w_{,i})_{,i}\, d\Omega - \int_\Omega u\, w_{,ii}\, d\Omega\right] = 0$$

$\Downarrow$ Div. theor.

$$\int_\Gamma w\, u_{,i}\, n_i\, ds - \int_\Gamma u\, w_{,i}\, n_i\, ds + \int_\Omega u\, w_{,ii}\, d\Omega = 0$$

or

$$\int_\Gamma w \overbrace{\left(\frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2\right)}^{q} ds - \int_\Gamma u \overbrace{\left(\frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2\right)}^{Q = q^*} ds$$

$$+ \int_\Omega u\, \nabla^2 w\, d\Omega = 0$$