Transformation of Stress

Theory & Examples

* **Triaxial** states of stress are shown in Fig. (1).

Only positive stresses on positive faces are shown

* **Biaxial** states of stress (Plane Stress):

When all stresses act in the same plane.

⇒ Work in 2-D as shown in Fig. (2) in the

x-y directions.

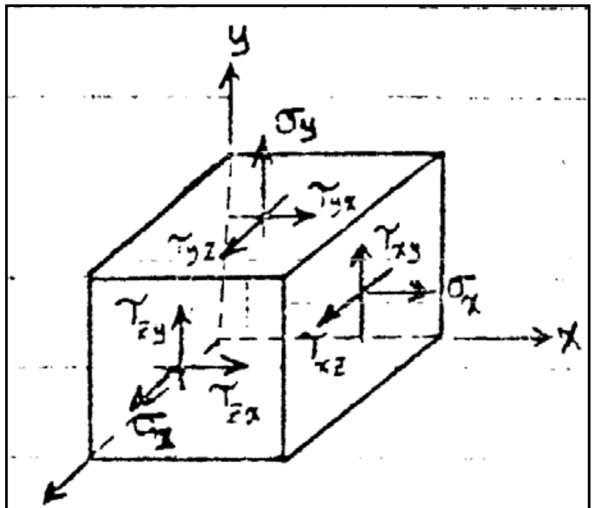


Fig. ①

It is possible to have the orientation in different directions, as shown in Fig. (3) in the ***x′-y′*** directions.

Now, relationships between the stresses in the ***x-y*** and ***x′-y′*** directions are sought.

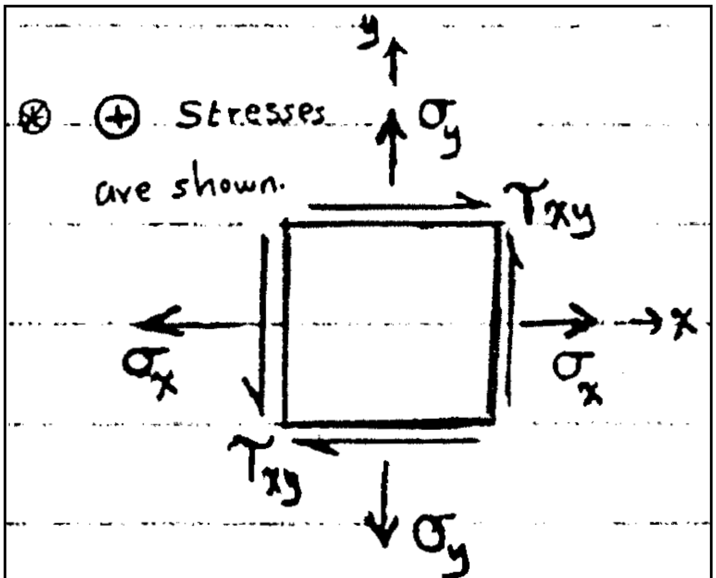


Fig. ②

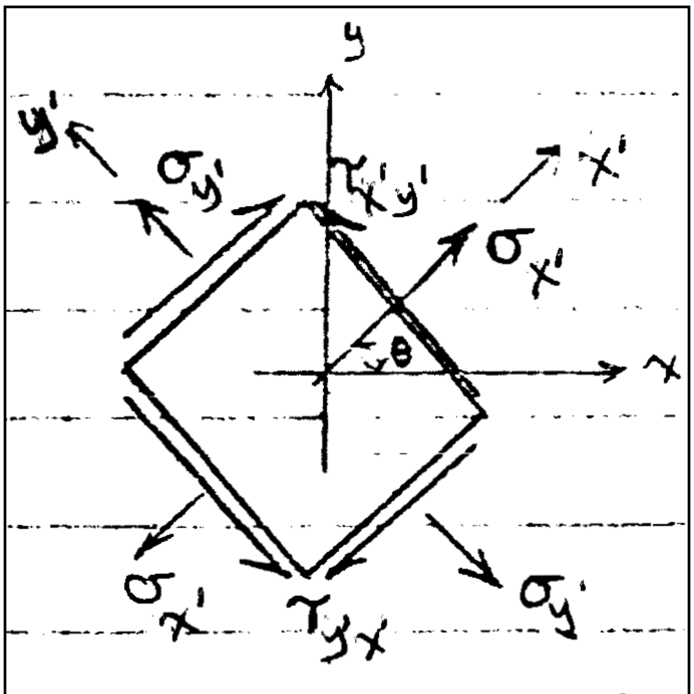


Fig. ③

From Fig. (4), the sum of **forces** in the x' and y' directions must be zero.

* ***Note that <u>forces</u> <u>not</u> <u>stresses</u> are added.***

(2 eqs. & 2 unkns.)

$\sum F_{x'} = 0$ & $\sum F_{y'} = 0$ $\Rightarrow$

$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$

$\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$

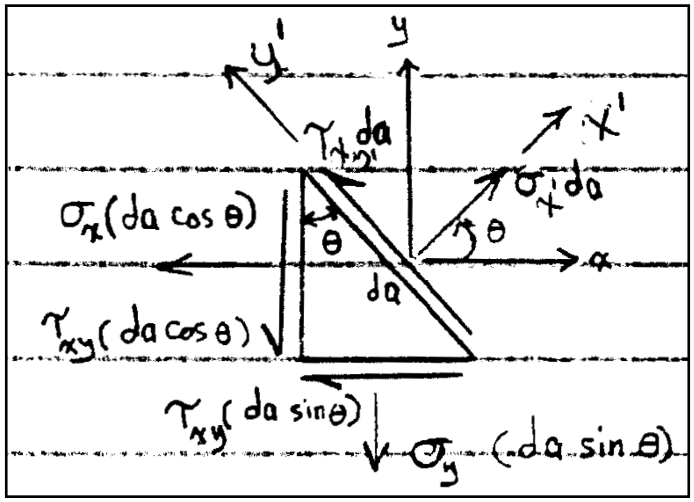


Fig. (4)

Recall that

$\cos^2\theta = (1 + \cos 2\theta)/2$

$\sin^2\theta = (1 - \cos 2\theta)/2$

$\sin\theta \cos\theta = \sin 2\theta/2$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\Rightarrow$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad [1]$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad [2]$$

Similarly,

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad [3]$$

$$\tau_{y'x'} = \tau_{x'y'} \quad [4]$$

* ***Note that $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'} = I$ (Invariant with <u>any</u> θ)*** $\Leftarrow$ **Use it to <u>check</u>.**

When σ & τ on <u>any</u> two orthogonal faces are known, the stress components on <u>all</u> (any) faces (plane stress) can be calculated.

Principal Normal Stresses:

$\sigma_{x'} = f(\theta) \Rightarrow$ *to get* $\sigma_{x'max}$, *set* $\frac{d\sigma_{x'}}{d\theta} = 0 \quad \Rightarrow \quad$ *find* $\theta_p \quad \Rightarrow \sigma_{x'max}$

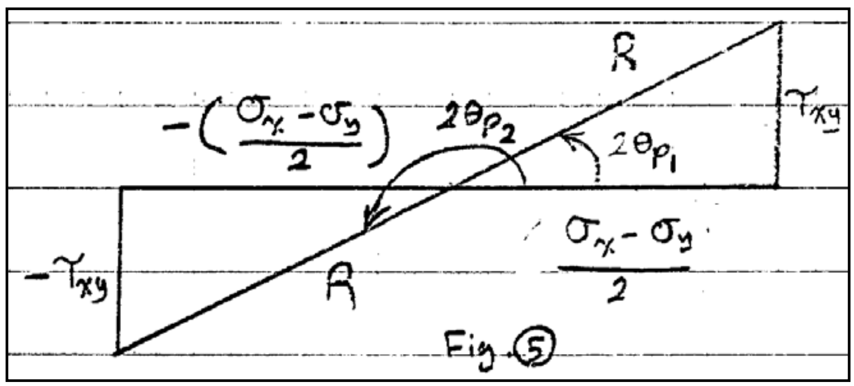
$$\frac{d\sigma_{x'}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta_p + 2\tau_{xy}\cos 2\theta_p = 0$$

Dividing by cos 2θ,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \quad [5]$$

From the equation above, Fig. (5) shown can be constructed.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad [6]$$



$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta_{p1} = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\sin 2\theta_{p2} = \frac{-\tau_{xy}}{R} \quad ; \quad \cos 2\theta_{p2} = \frac{-(\sigma_x - \sigma_y)/2}{R}$$

$$\Rightarrow \boxed{\sigma_{\substack{max \\ min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_x + \sigma_y}{2} \pm R} \quad [7]$$

⇑ **Principal Normal Stresses**

The directions are given by θ_{p1} and θ_{p2}

Note that $2\theta_{p2} = 2\theta_{p1} + \pi \quad \Rightarrow \quad \theta_{p1} \perp \theta_{p2}$

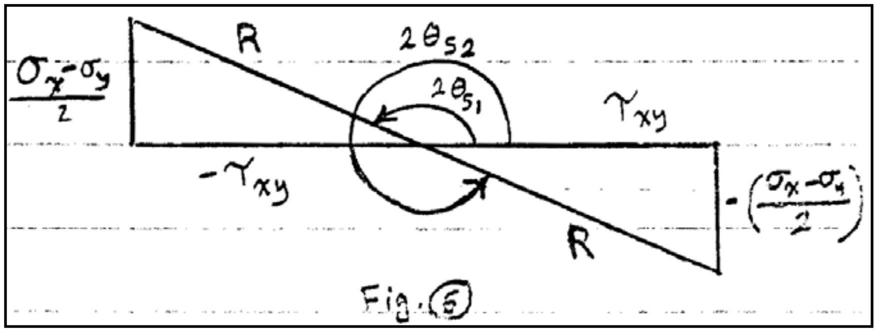
Also note that $\tau_{x'y'} = 0$ ***on the planes which the principal normal stresses act.***

maximum Shear Stresses: $\Leftarrow$ sometimes called principal τ

$\tau_{x'y'} = f(\theta)$ $\Rightarrow$ *The value of θ_s can be obtained by setting* $\dfrac{d\tau_{x'y'}}{d\theta} = 0$.

$$\Rightarrow \quad \tan 2\theta_s = \frac{-\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}} \qquad [8]$$

[Fig. (6)]



There are 2 possible values for θ_s

$$\Rightarrow \quad \sin 2\theta_{s1} = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\cos 2\theta_{s1} = \frac{-\tau_{xy}}{R}$$

$$\sin 2\theta_{s2} = \frac{-(\sigma_x - \sigma_y)/2}{R} \quad ; \quad \cos 2\theta_{s2} = \frac{\tau_{xy}}{R}$$

$$\Rightarrow \quad \boxed{\tau_{\substack{max\\min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm R} \qquad [9]$$

$\Uparrow$ **Maximum (Principal) Shear Stresses**

$$\boxed{\sigma_{x'} = \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2}} \qquad [10]$$

$\Uparrow$ ***Normal Stresses on the Planes of Maximum (Principal) Shear Stresses***

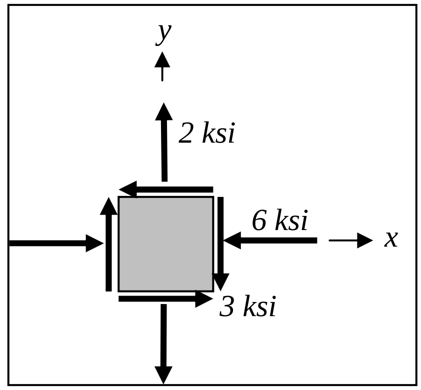
Note that $2\theta_{s2} = 2\theta_{s1} + \pi \quad \Rightarrow \quad \theta_{s1} \perp \theta_{s2}$

Also note that $\tan 2\theta_p$ is the negative reciprocal of $\tan 2\theta_s$: $\tan 2\theta_p = -1/\tan 2\theta_s$

$\Rightarrow 2\theta_s = 2\theta_p + \pi/2 \quad \Rightarrow \quad \theta_s = \theta_p + 45^o$ **[11]**

Thus, there is a 45°-angle between the planes of principal normal and maximum shear stresses.

Example 1:



Given:

The state of stress shown

Req'd:

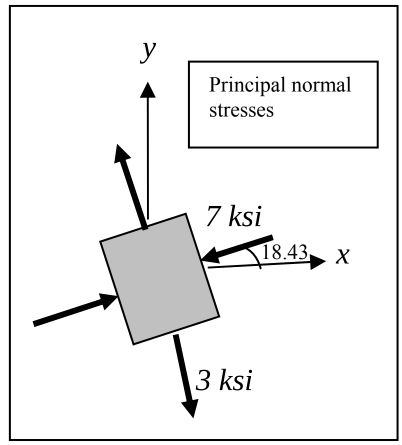
a) The principal stresses & directions

b) σ & τ associated with an element oriented 10^o cw of the element shown.

Show the results on properly oriented elements.

Use the equations for the solution.

Solution:



a)

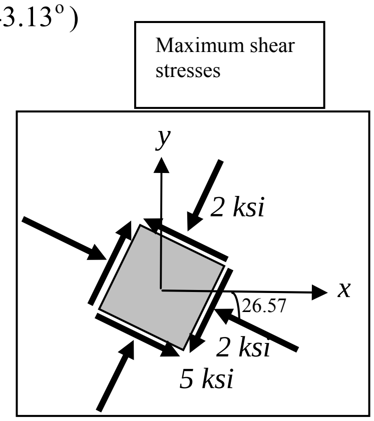
$$\sigma_{\substack{max\\min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-6+2}{2} \pm \sqrt{\left(\frac{-6-2}{2}\right)^2 + (-3)^2} = -2 \pm 5$$

$\Rightarrow$ **σ_{max} = 3 ksi ; σ_{min} = -7 ksi**

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-3)}{-6-2} = 0.75 \Rightarrow 2\theta_{p1} = 36.87^o \ , \ 2\theta_{p2} = 216.87^o \ (-143.13^o)$$

$\Rightarrow$ **θ_{p1} = 18.43^o ; θ_{p2} = 108.43^o (-71.57^o)**



To see θ_{p1} corresponds to σ_{max} or σ_{min}, substitute θ in Eq. [1] by $\theta_{p1} = 18.4349^o$

$$\Rightarrow \sigma_{x'} = \frac{-6+2}{2} + \frac{-6-2}{2}\cos 36.87^o - 3\sin 36.57^o = -7\text{ ksi}$$

$\Rightarrow$ **θ_{p1} is the direction of σ_{min} as shown.**

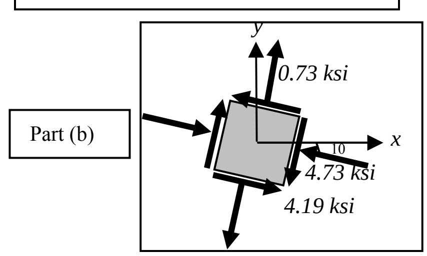
$\tau_{\substack{max\\min}} = \pm R \Rightarrow$ **τ_{max} = 5 ksi ; τ_{min} = -5 ksi** $\quad$ $\sigma_{x'} = \sigma_{y'} = \frac{-6+2}{2} = -2$ ksi

$\tan 2\theta_s = -1/0.75 \Rightarrow$ **θ_{s1} = -26.57^o ; θ_{s2} = 63.43^o** ; $\tau(-26.57) = -5$ ksi $= \tau_{min} \leftrightarrow \theta_{s1}$

b) From. Eqs. [1] to [4], **$\sigma_{x'}$ (-10^o) = -4.73 ksi ; $\sigma_{y'}$ (-10^o) = 0.73 ksi**

Note that $\sum \sigma_i$ = - 4 (always)

$\tau_{x'y'}$ (-10^o) = $\tau_{y'x'}$ (-10^o) = -4.19 ksi ($\pm$)



Mohr's Circle:

The general equation of the circle is

$$(x-a)^2 + (y-b)^2 = R^2 \qquad [12]$$

By rearranging equation [1] and squaring both sides of equations [1] & [2], and then adding, the following equation is obtained:

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad [13]$$

By comparing eq. [13] with eq. [12], it can be seen that

$x = \sigma_{x'}$

$a = (\sigma_x + \sigma_y)/2 = \sigma_{average}$

$y = \tau_{x'y'}$

$b = 0$

$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Thus, by constructing a circle with the properties above, and by referring to Fig. (5), the values of $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ with any θ can be found; this includes the principal stresses and their directions. This circle is called Mohr's Circle because Mohr brought the idea of such a circle. It has several applications, other than stresses.

<u>Steps for constructing Mohr's circle and determining the principal stresses & directions and σ & τ with any θ:</u>

(1) Draw the σ-τ axes in the x-y (i.e., horizontal-vertical) directions with "appropriate" scale. Note the direction of $+\tau$ (down).

(2) Put the points x (σ_x, τ_{xy}) and y (σ_y , -τ_{xy}) on the figure.

(3) Connect the two points x and y by a straight line. The point of intersection of the line xy and the σ-axis is the center of the circle C, and cx & cy are two radii of such a circle.

(4) Construct the circle with C as its center and Cx (or Cy) the radius.

(5) The points of intersection of the circle and the σ-axis are the principal normal stresses; the one to the right is the maximum, and the one to the left is the minimum.

(6) The radius of the circle is τ_{max}, and τ_{min} = - R.

(7) The angle measured <u>from the x-axis to σ_{max}</u> gives $2\theta_{p1\ (or\ p2)}$ (i.e., <u>double</u> the angle), and the angle <u>measured from x to σ_{min}</u> is $2\theta_{p2\ (or\ p1)}$.

(8) The angle measured <u>from x to τ_{max}</u> is $2\theta_{s1\ (or\ s2)}$ and the angle <u>from x to τ_{min}</u> is $2\theta_{s2\ (or\ s1)}$.

(9) The similar triangles show in Fig. (7) are used to calculate the required values (stresses and their directions).

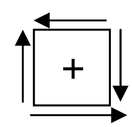
(10) To calculate the stresses on planes oriented θ^o from the x-axis on the real plane, go <u>2</u>θ <u>from the x-axis</u> on the imaginary plane (Mohr's circle), and then draw a straight line passing through C. Use the triangles shown in Fig. (7) to determine the stress values.

<u>Remember:</u>

* <u>Start from x</u>, <u>Double the angle</u>

* **Some books use *$+\tau$ as up (not down as done here). If it is so, then all angles will be in the* <u>opposite</u> directions. You can go in the same direction (*not the opposite*) if one of the following is done:**

(1) Reverse the "sign convention" for shear. $\Rightarrow$



(2) Plot x(σ_x , -τ_{xy}) , y (σ_y , τ_{xy}).

(3) Plot + τ axis down (as done here).

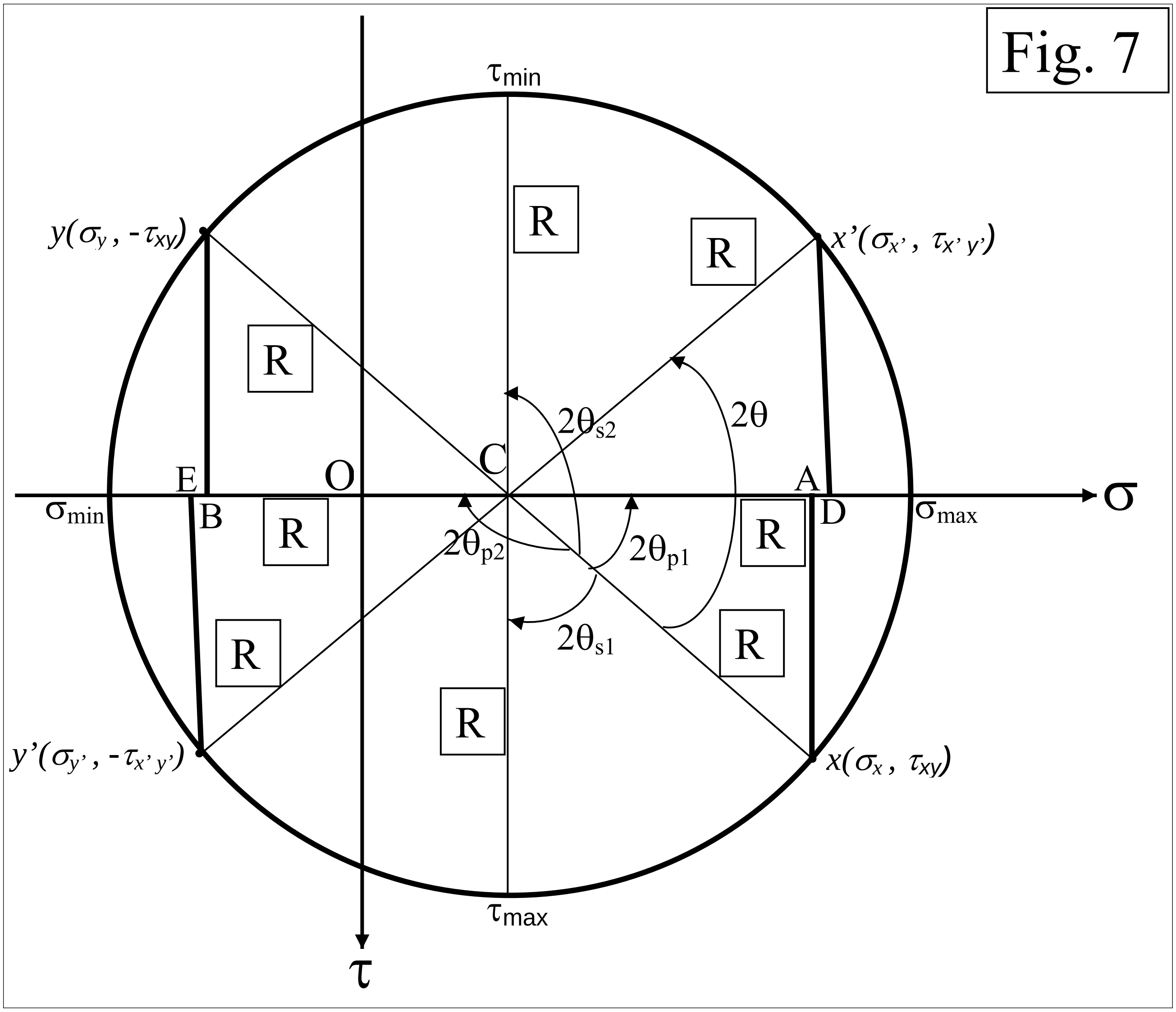


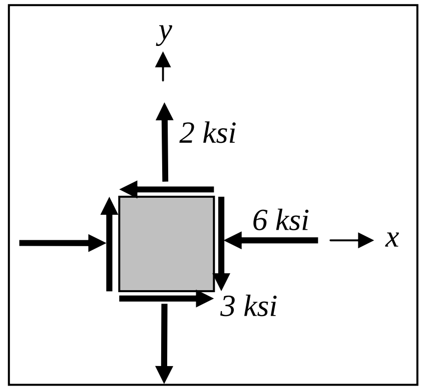
Fig. 7
τmin
R
y(σy , -τxy)
R
x'(σx' , τx' y')
R
2θs2
2θ
C
E
O
A
σ
σmin
B
R
2θp2
2θp1
R
D
σmax
2θs1
R
R
R
y'(σy' , -τx' y')
x(σx , τxy)
τmax
τ

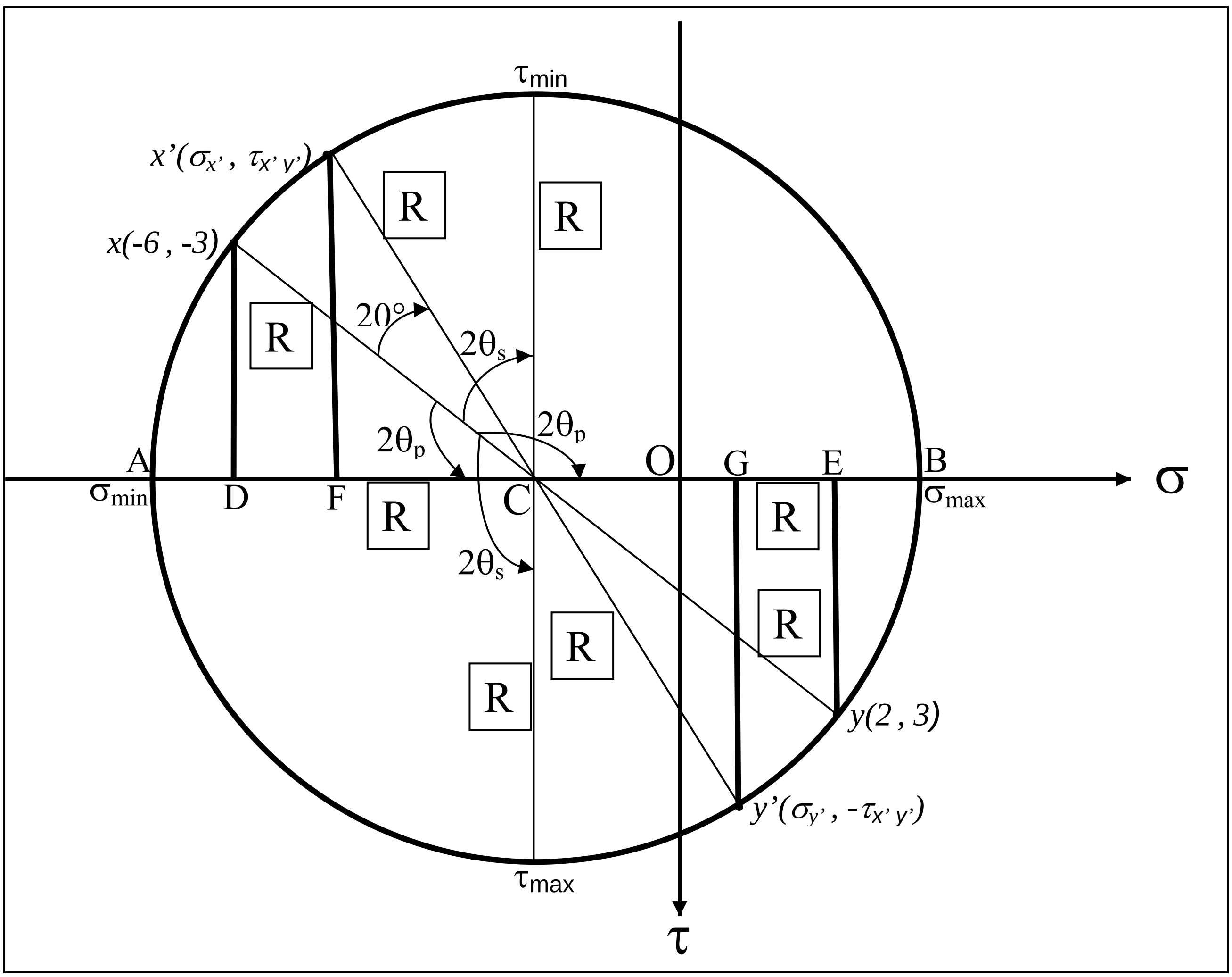
<u>Example 2</u>:

Rework Example 1 using Mohr's circle.

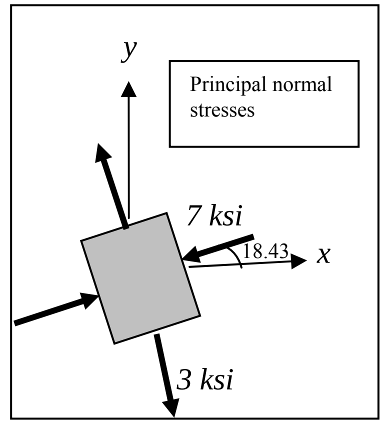
Solution:

a) The steps given above will be followed.





$$OC = \frac{OE + OD}{2} = \frac{2-6}{2} = -2$$



Using the triangle CxD or CyE, R and $2\theta_p$ can be calculated. ⇒

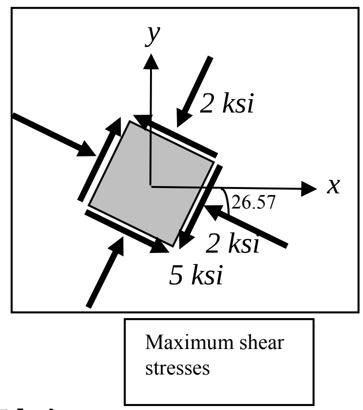
$$\tan 2\theta_{p1} = \frac{xD}{CD} = \frac{|xD|}{|OD| - |OC|}$$

$$= \frac{3}{6-2} = 0.75$$

⇒ $2\theta_{p1} = 36.87^o$ ⇒

⇒ θ_{p1} = 18.43º ccw (↺) measured from the x-axis to the axis of σ_{min}

* Remember: **Double the angle** **measured from x**



⇒ $2\theta_{p2} = 180^o - 2\theta_{p1}$

⇒ θ_{p2} = 71.57º CW (↻) measured from x to σ_{max}

R = Cx = Cy = |xD| /sin $2\theta_{p1}$ = 3/sin 36.87º = **5**

σ_{max} = OB = CB - |OC| = R - |OC| = 5-2 ⇒ **σ_{max} = 3 ksi**

σ_{min} = - |OA| = - (|OC| + |CA|) = - (|OC| + R) = - (2 + 5) ⇒ **σ_{min} = -7 ksi**

⇑

Take care of the signs by inspection.

$\tau_{\substack{max \\ min}} = \pm R$ ⇒ **τ_{max} = 5 ksi** ; **τ_{min} = - 5 ksi**

$2\theta_{s1} = 90^o - 2\theta_{p1} = 90 - 36.87^o$ ⇒ **θ_{s1} = 26.57º cw (↻) measured from x to τ_{min}**

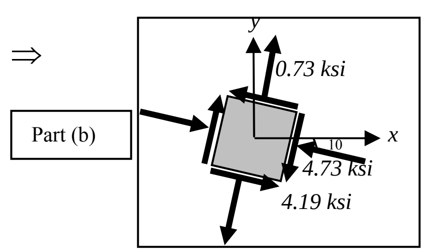
$2\theta_{s2} = 90^o + 2\theta_{p1} = 90 + 36.87^o$ ⇒ **θ_{s2} = 63.43º ccw (↺) measured from x to τ_{max}**

b) $\sigma_{x'}$ (-10º) = OF = - (|OC| + |CF|) = - [|OC| + R cos ($2\theta_{p1}$ + 20º)] = - [2 + 5 cos (56.87º)] ⇒

$\sigma_{x'}$ (-10) = -4.73 ksi

$\sigma_{y'}$ (-10º) = OG = R cos ($2\theta_{p1}$ + 20º) - |OC| = 5 cos (56.87) - 2 ⇒

$\sigma_{y'}$ (-10º) = 0.73 ksi

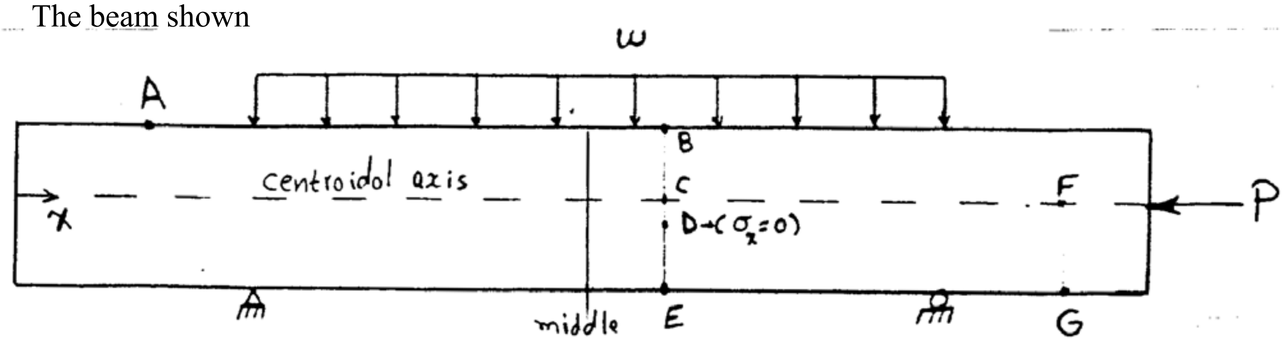


$\tau_{x'y'} = \pm R \sin (56.87^o)$ ⇒ **$\tau_{x'y'}$ (-10º) = ± 4.19 ksi**

Example 3:

Given:

The beam shown

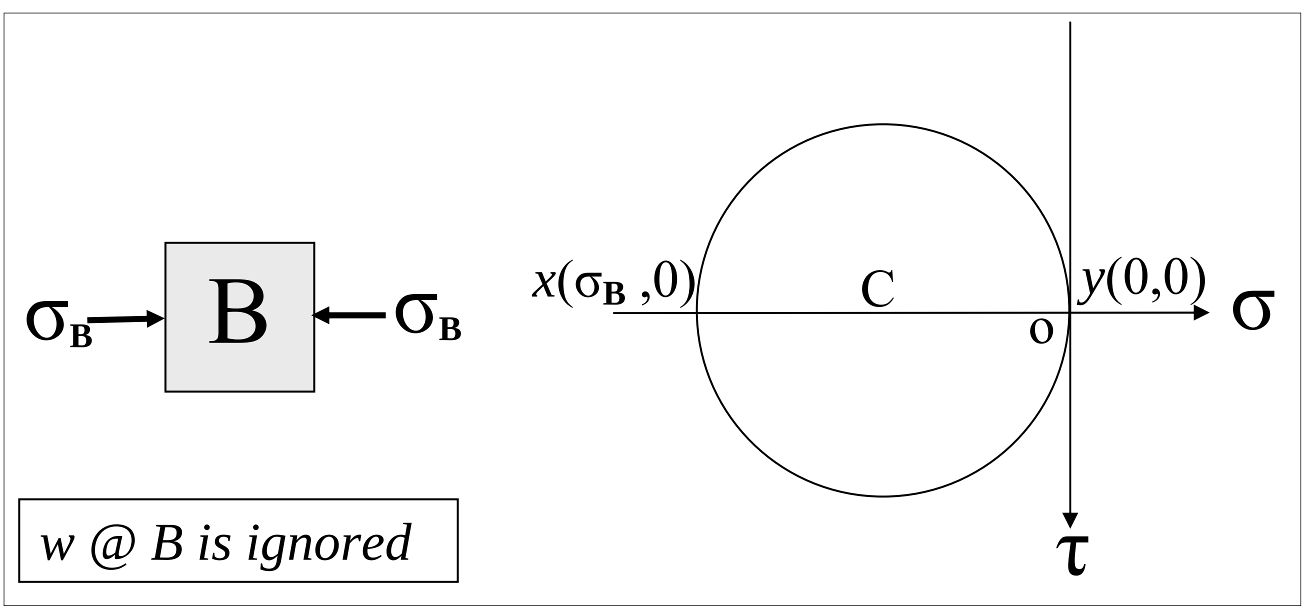


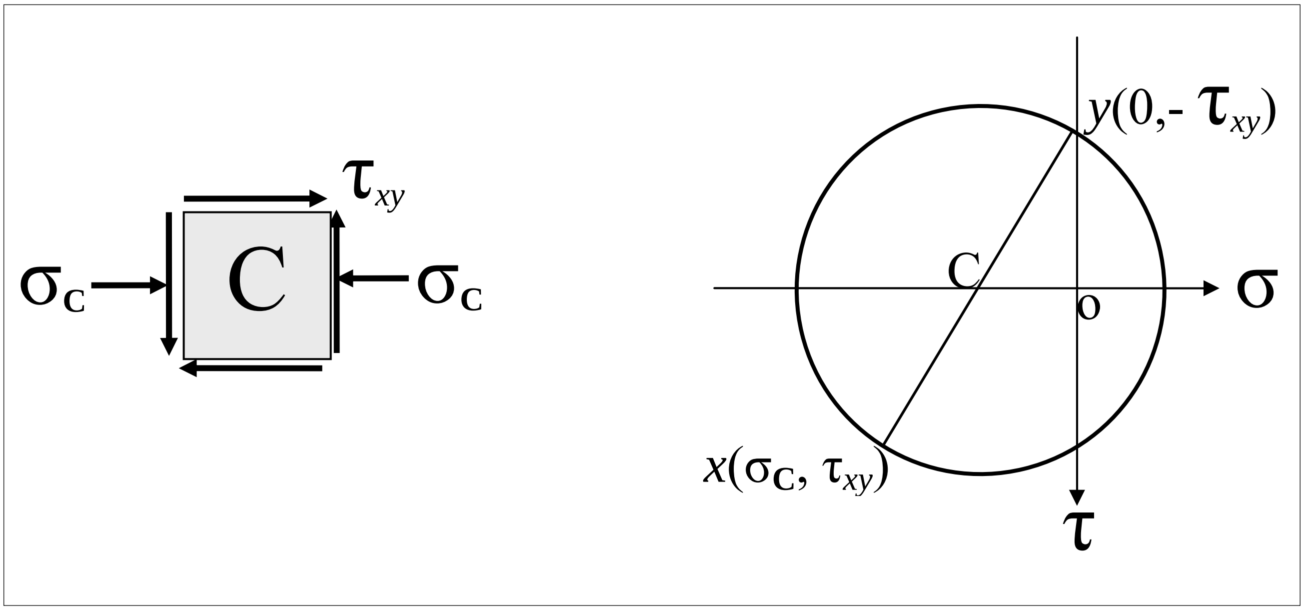
Req'd.:

Qualitatively, sketch the **state of stress** and **Mohr's circle** for each of the **points A to G**.

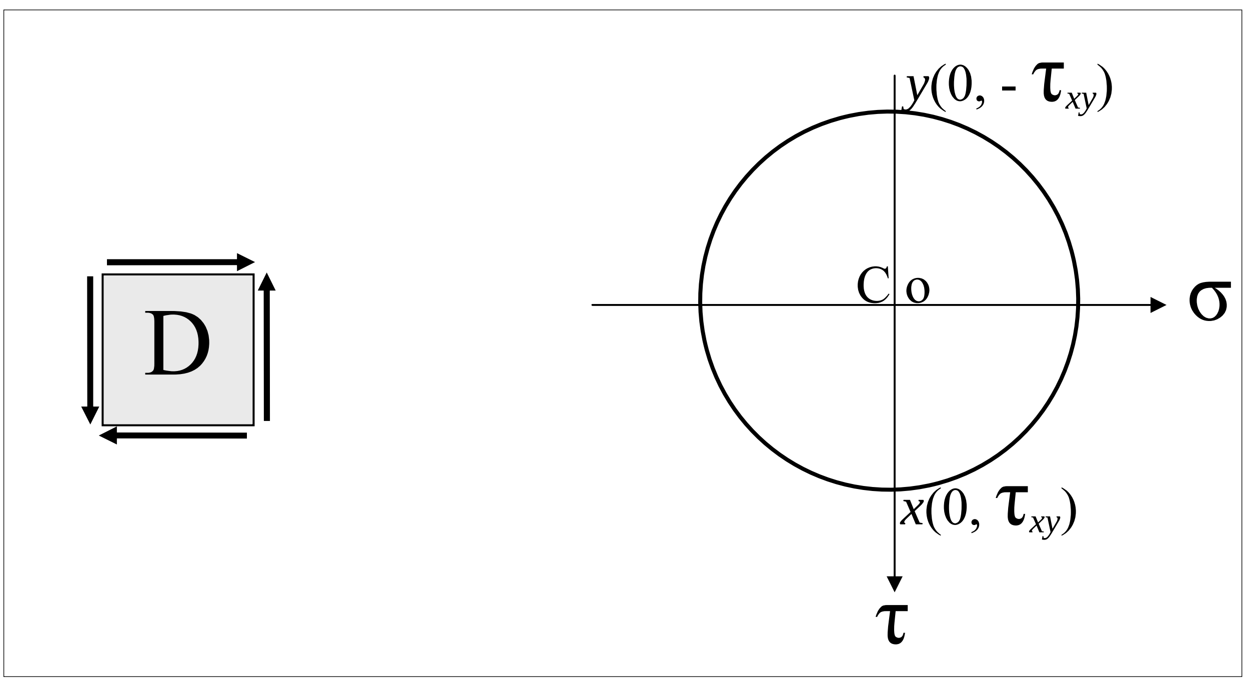
Solution:

Note that σ_y is always assumed ***zero*** *(ignored) in* ***beams****.*

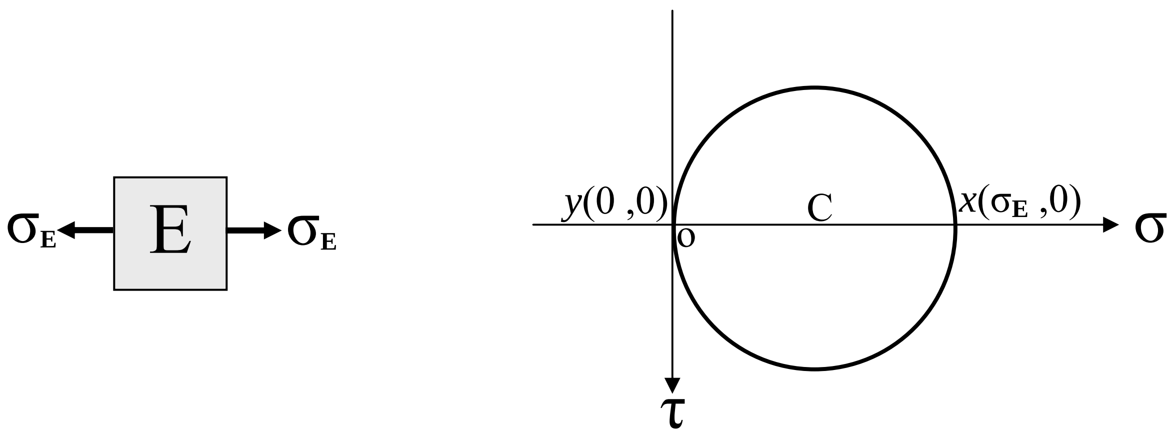


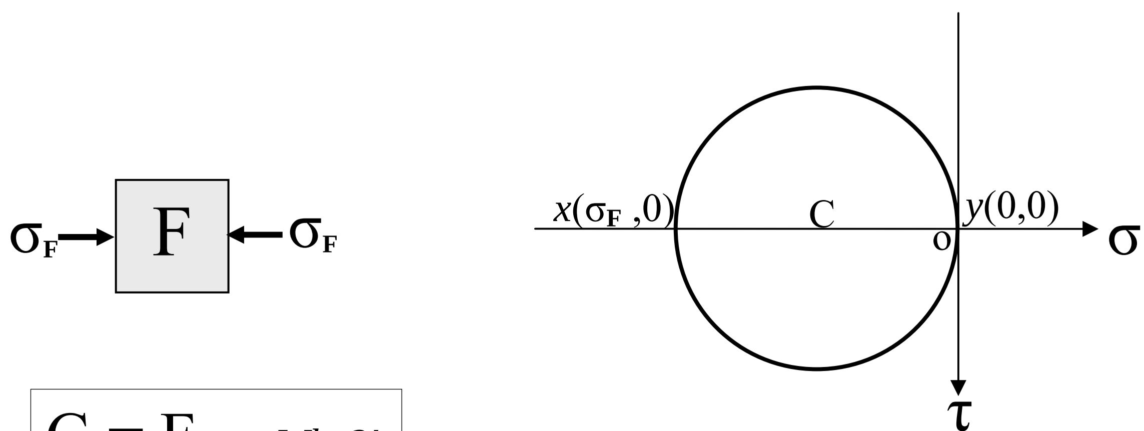


σC
C
σC
τxy
y(0,- τxy)
C
o
σ
x(σC, τxy)
τ



D
y(0, - τxy)
C o
σ
x(0, τxy)
τ



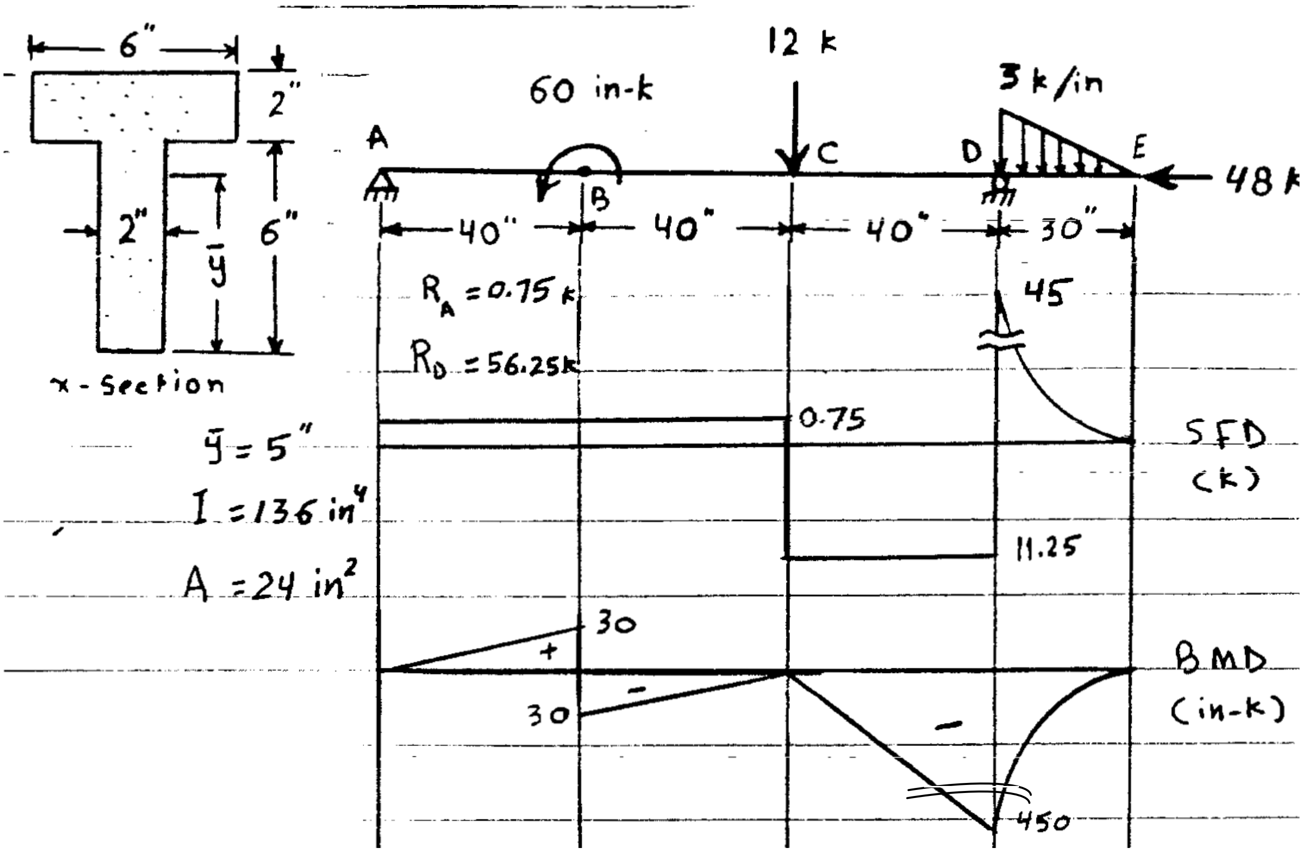


G = F *Why?!*

Example 4:

Given:

The beam shown



Req'd.:

The principal normal and shear stresses and their directions at the point(s) of maximum stresses.

Sol'n.:

M_{max} = 450 in-k @ D

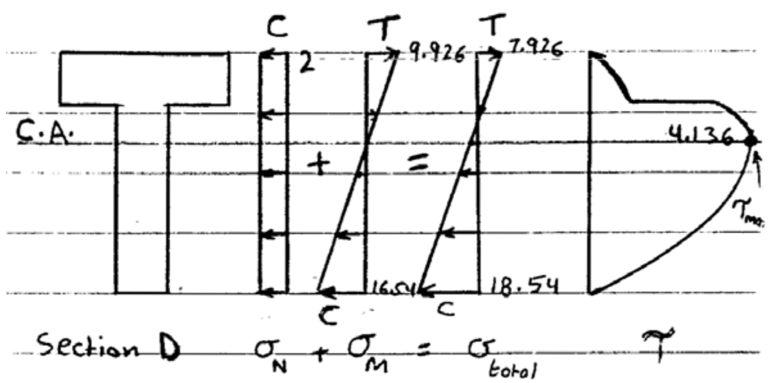
This M will give σ_{max} (both T & C)

V_{max} = 45 k @ D also

$$\sigma = \sigma_N + \sigma_M = \pm \frac{N}{A} \pm \frac{My}{I}$$

$$\sigma_{top} = \frac{-48}{24} + \frac{450(3)}{136} = -2 + 9.926$$

$$= \mathbf{7.926\ ksi\ \ (T)}$$



$$\sigma_{\text{bottom}} = -\frac{48}{24} - \frac{450(5)}{136} = -2 - 16.54$$

$= \mathbf{18.54\ ksi \quad (C)}$

$$\sigma_{\text{C.A.}} = \frac{-48}{24} + 0 = 2 \text{ ksi} \quad \text{(C)}$$

$\tau_{top} = \tau_{bottom} = 0$

$\tau_{C.A.} = \boldsymbol{\tau_{max}} = \dfrac{VQ}{Ib} = 45\ (5\times2\times2.5)/136(2) = \mathbf{4.136\ ksi}$

σ's above are all σ_x.

$\sigma_y = 0$ at all points (always the case in beam theory)

We need to calculate the principal stresses at 3 points (top, bottom, and N.A. of section D).
⇒ Choose the maximum normal (T & C) and shear stresses.

1) Top of D : (**Mohr's circle** is used for the max/min stress values below, *not shown*)

   $\sigma_x = 7.926$ ksi , $\sigma_y = 0$, $\tau_{xy} = 0$

   ⇒ $\sigma_{max} = 7.926$ ksi , $\sigma_{min} = 0$, $\tau_{max} = 3.963$ ksi , $\tau_{min} = -3.963$ ksi

2) Bottom of D: $\sigma_x = -18.54$ ksi , $\sigma_y = 0$, $\tau_{xy} = 0$

   ⇒ $\sigma_{max} = 0$, $\sigma_{min} = -18.54$ ksi , $\tau_{max} = 9.27$ ksi , $\tau_{min} = -9.27$ ksi

3) Centroidal Axis: $\sigma_x = -2$ ksi , $\sigma_y = 0$, $\tau_{xy} = 4.136$ ksi

   ⇒ $\sigma_{max} = 3.2$ ksi , $\sigma_{min} = -5.2$ ksi , $\tau_{max} = 4.2$ ksi , $\tau_{min} = -4.2$ ksi ⇒

$\boldsymbol{\sigma^T_{max} = 7.926\ ksi}$	**@ D (Top)**
$\boldsymbol{\sigma^C_{max} = 18.54\ ksi}$	**@ D (Bottom)**
$\boldsymbol{\tau_{max} = 9.27\ ksi}$	**@ D (Bottom)**

* ***Important note:* When τ_{xy} is zero at a point, then σ_x and σ_y are themselves principal stresses at that particular point (as seen above).**