#21

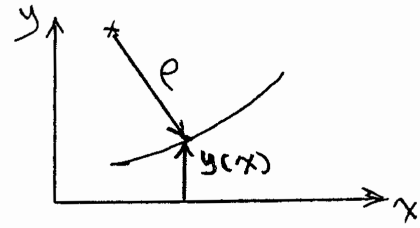
Deflection of Beams
(Theory & Examples)

* Moment – Curvature Relation (developed earlier):

$$\frac{1}{\rho} = \frac{M}{EI}$$

From calculus, the curvature of the plane curve shown is given by

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

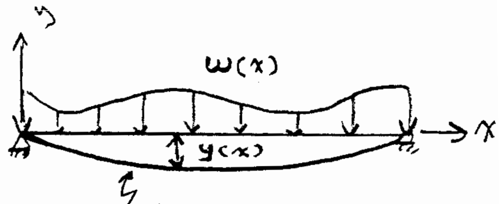


For "very small" deformation (as it is the case in most engineering problems), $(dy/dx)^2 << 1$

Thus,

$$\frac{1}{\rho} \simeq \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{M}{EI} = \frac{d^2y}{dx^2} \quad \Leftarrow \quad \underline{\underline{y}} \text{ is the } \underline{\underline{\text{deflection}}}$$



elastic curve (deformed shape)

$$\Rightarrow \quad M = EI \frac{d^2y}{dx^2}$$

Recall that $V(x) = \frac{dM}{dx}$ & $w(x) = dV/dx$

Thus, the summary is

$$\boxed{w(x) = \frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = \text{load}} \quad ① \qquad \boxed{V(x) = \frac{dM}{dx} = EI\frac{d^3y}{dx^3} = \text{shear}} \quad ②$$

$$\boxed{M(x) = EI\frac{d^2y}{dx^2} = \text{Moment}} \quad ③ \qquad \boxed{\theta(x) = \frac{dy}{dx} = \text{slope}} \quad ④$$

$\Rightarrow$

$V = \int w\,dx$	$M = \int V\,dx$	$\theta = \int \frac{M}{EI}\,dx$	$y = \int \theta\,dx$
	$= \iint w\,dx\,dx$	$= \iiint \frac{w}{EI}\,dx\,dx\,dx$	$= \iiiint \frac{w}{EI}\,dx\,dx\,dx\,dx$

The deflection of the beam is needed for two main reasons:

1) To limit the maximum deflection (i.e. $y_{max} \leq y_{allowable}$)

2) To determine the reactions in statically indeterminate problems

If the beam is designed based on the maximum allowable deflection, this is called "design for stiffness". If the design is based on limiting the maximum (allowable) stress, it is called "design for strength". In most applications, the stress controls (i.e. limiting the stress is more important than limiting the deflection because deflections are usually "very small" in "typical" structures). Thus, the second reason above is more important than the first one.

There are many methods for calculating slopes and deflections of beams. In this course, only three methods are covered. In Structural analysis I (CE 305), several methods, including energy and computer procedures, are discussed in details. The three are

1) Double Integration
2) Successive Integration
3) Singularity function

In fact, these three methods have the same theoretical basis; thus, they could be considered as one way, with different branches, for determining deflections. It is the elementary, fundamental, or basic method of integration.

The deflection due to the moment only will be discussed here. The deflection due to the shear is discussed in CE 305 and other courses. However, y_V is usually much less than y_M. Therefore, y_V is negligible in most cases.

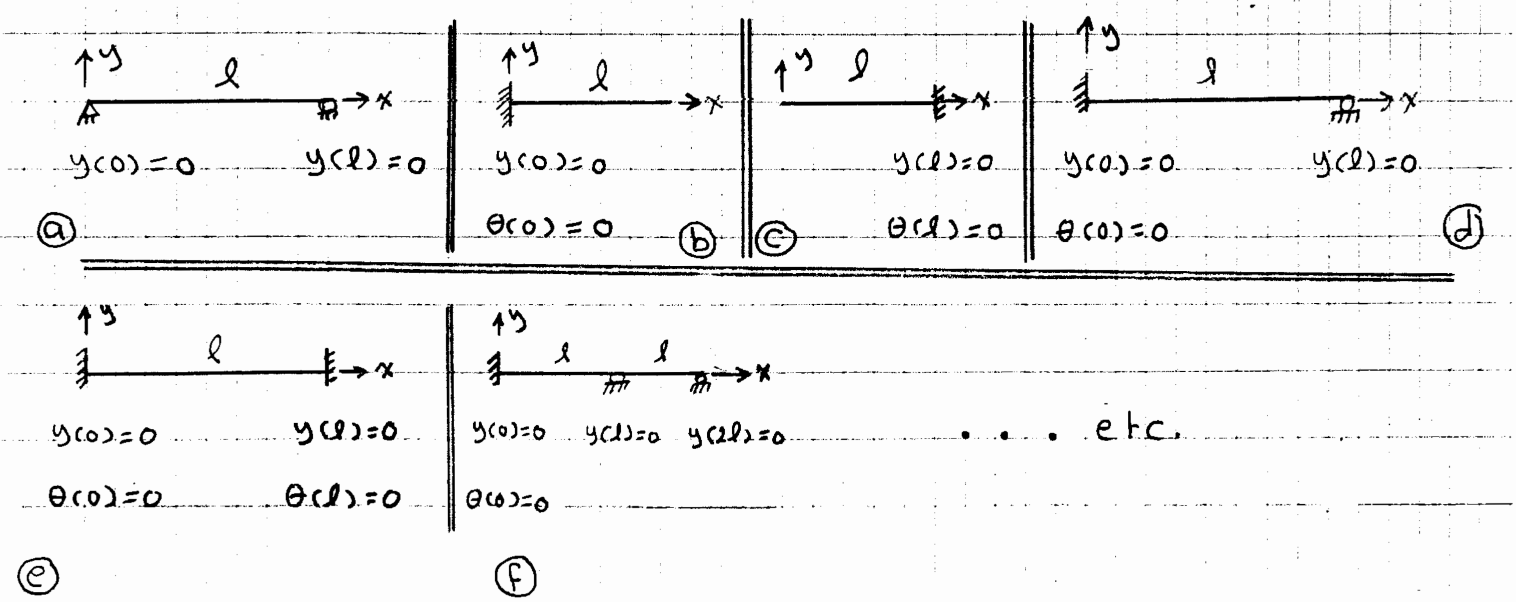
i) Double integration method :

If the moment equation is known or it can be obtained easily, then by integrating twice (double), the deflection equation can be determined. In this case, two integration constants for each moment equation appear; therefore, two boundary conditions (B.C.s) for each equation are needed. Note that there could be more than one moment equation in a beam, depending on the loading conditions.

In statically indeterminate beams, the moment equation can not be written explicitly, but it must be written in terms of some of the unknown reactions. Thus, more than two boundary conditions are needed in order to solve for the two constants and the unknown reactions, as will be seen in the examples. In general, the number of B.C.s has to equal to 2 plus the degree of statical indeterminacy of the beam (n), or

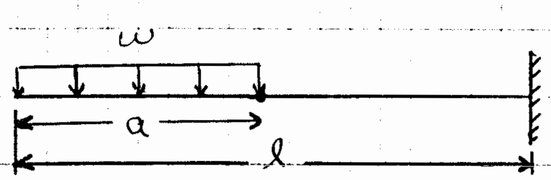
$$\text{B.C.s} = 2 + n$$

Examples of B.C.s are shown below (# 1ⓐ to 1ⓕ)



The discussion above about B.C.s. is true for beams with a single moment equation. If the beam has more than one moment equation, then the total number of constants is equal to 2 times the number of equations. Thus, two B.C.s are not enough to solve for all the constants. Therefore, the concept of continuity conditions is introduced. That is, the slope and deflection must be continuous between adjacent intervals. These continuity conditions give additional or supplementary equations which make it possible to solve for all the constants, as illustrated below. However, as the number of moment equations increases the number of unknown constants increases as well, giving a large number of equations which have to be solved simultaneously. This could be very tedious and time-consuming; thus, this method becomes impractical, and a better one, called singularity function method, is introduced, as will be discussed latter. Because of that, beams with one moment equation only are covered by this method as well as by the method of successive integrations.

Examples of Continuity conditions (C.C.s.): #2 ⓐ and 2 ⓑ

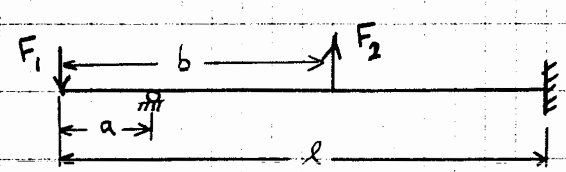


$y(a-) = y(a+)$ $\quad$ $y(l) = 0$

$\theta(a-) = \theta(a+)$ $\quad$ $\theta(l) = 0$

$\Uparrow$ C.C.s. $\quad$ $\Uparrow$ B.C.s.

ⓐ



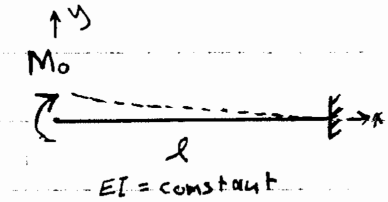
$y(a) = 0$ $\quad$ $y(b-) = y(b+)$ $\quad$ $y(l) = 0$

$\theta(a-) = \theta(a+)$ $\quad$ $\theta(b-) = \theta(b+)$ $\quad$ $\theta(l) = 0$

$\Uparrow$ C.C.s. & B.C.s. $\quad$ $\Uparrow$ C.C.s. $\quad$ $\Uparrow$ B.C.s.

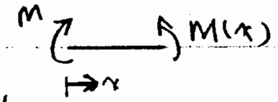
ⓑ

Example 3:



Derive an expression for the elastic curve (deflection) and find the maximum y & θ in the beam shown.

Solution:



From the FBD shown, the moment equation can be written:

$M(x) = M_o$

$EI\,\theta(x) = \int M(x)\,dx = \int M_o\,dx = M_o x + C_1$

$EI\,y(x) = \int EI\theta(x)\,dx = \int (M_o x + C_1)\,dx = \frac{1}{2} M_o x^2 + C_1 x + C_2$

B.C.s. : $y(l) = 0$; $\theta(l) = 0$ [two B.C.s. and two constants ⇒ ok]

$\theta(l) = 0 \Rightarrow M_o l + C_1 = 0 \Rightarrow C_1 = -M_o l$

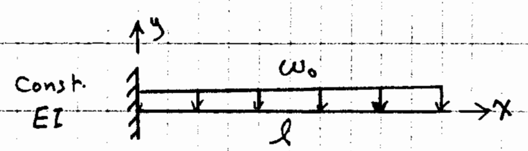
$y(l) = 0 \Rightarrow \frac{1}{2} M_o l^2 + (-M_o l) l + C_2 = 0 \Rightarrow C_2 = \frac{1}{2} M_o l^2$

$\Rightarrow \underline{EI\theta(x) = M_o (x - l)}$; $\underline{EI y(x) = M_o \left(\frac{x^2}{2} - l x + \frac{l^2}{2}\right)}$

θ_{max} & y_{max} are at the free end. y_{max} is always at the free end or at the point where $\theta = \frac{dy}{dx} = 0$

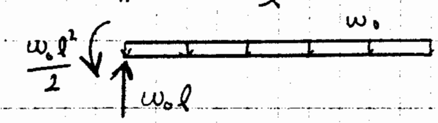
$$\left.\begin{aligned} \theta_{max} &= \theta(0) \Rightarrow \underline{\theta_{max} = -M_o l = M_o l \ (\text{cw})} \\ y_{max} &= y(0) \Rightarrow \underline{y_{max} = M_o l^2/2 \quad (\uparrow)} \end{aligned}\right\} @\ \underline{x = 0}$$

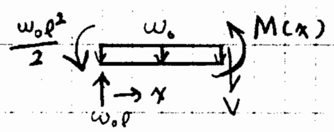
Example 4:



Derive equations for θ and y for the beam shown.

Solution:





$M(x) = w_o l x - \frac{1}{2} w_o l^2 - w_o x^2/2$

$EI\theta = \int M\,dx = \int (w_o l x - \frac{1}{2} w_o l^2 - w_o \frac{x^2}{2})\,dx$

$= \frac{1}{2} w_o l x^2 - \frac{1}{2} w_o l^2 x - \frac{1}{6} w_o x^3 + C_1 = -\frac{1}{6} w_o x^3 + \frac{1}{2} w_o l x^2 - \frac{1}{2} w_o l^2 x + C_1$

$EIy = \int EI\theta\,dx = \int (-\frac{1}{6} w_o x^3 + \frac{1}{2} w_o l x^2 - \frac{1}{2} w_o l^2 x + C_1) = -\frac{1}{24} w_o x^4 + \frac{1}{6} w_o l x^3 - \frac{1}{4} w_o l^2 x^2 + C_1 x + C_2$

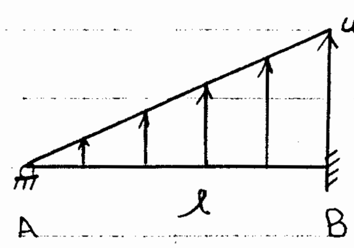
B.C.s. : $\theta(0) = 0 \Rightarrow C_1 = 0$ and $y(0) = 0 \Rightarrow C_2 = 0 \Rightarrow$

$\underline{\theta(x) = \frac{w_o}{6EI}(-x^3 + 3 l x^2 - 3 l^2 x)}$; $\underline{y(x) = \frac{w_o}{24EI}(-x^4 + 4 l x^3 - 6 l^2 x^2)}$

Solve this example with x from right to left ($x \leftarrow$). Which one is easier ?! Why ?

Example 5:

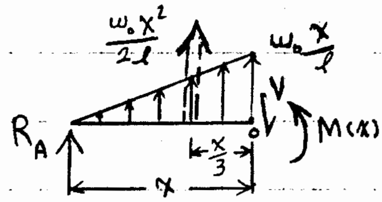
Determine the reactions of the beam shown.



EI = constant

Solution:

Since the beam is statically indeterminate, the reactions are not known and, thus, the moment equation can not be written explicitly; therefore, it has to be written in terms of the unknown reactions. ⇒



$$\overset{+}{\circlearrowleft} \Sigma M_o = 0 \Rightarrow M(x) - R_A x - \left(\frac{w_o x^2}{2l}\right)\left(\frac{x}{3}\right) \Rightarrow$$

$$M(x) = R_A x + \frac{w_o}{6l} x^3$$

$$EI\theta(x) = \int M(x)dx = \int \left(R_A x + \frac{w_o}{6l}x^3\right)dx = \frac{1}{2}R_A x^2 + \frac{1}{24l} w_o x^4 + C_1$$

$$EI\,y(x) = \int EI\theta(x) dx = \int \left(\frac{1}{2}R_A x^2 + \frac{1}{24l} w_o x^4 + C_1\right)dx = \frac{1}{6}R_A x^3 + \frac{1}{120l} w_o x^5 + C_1 x + C_2$$

B.C.s.: $y(0) = 0$; $\theta(l) = 0$; $y(l) = 0$ ⇒ 3 B.C.s. & 3 unknowns (C_1, C_2, R_A) ⇒ ok

$y(0) = 0 \Rightarrow C_2 = 0$

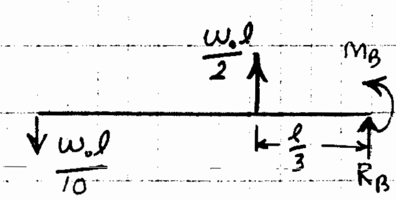
$\theta(l) = 0 \Rightarrow \frac{1}{2}R_A l^2 + \frac{w_o}{24l} l^4 + C_1 = 0 \Rightarrow \frac{1}{2}l^2 R_A + C_1 + \frac{w_o}{24} l^3 = 0$ ①

$y(l) = 0 \Rightarrow \frac{1}{6}R_A l^3 + \frac{w_o}{120l} l^5 + C_1 l = 0 \Rightarrow \frac{1}{6}l^2 R_A + C_1 + \frac{w_o}{120} l^3 = 0$ ②

By Solving Eqs. ① & ②, $R_A = -\frac{w_o l}{10} \Rightarrow \underline{R_A = \frac{w_o l}{10}\ (\downarrow)}$ and $C_1 = \frac{w_o l^3}{120}$

$$\Rightarrow \underline{\theta(x) = \frac{w_o}{120 l EI}\left(5x^4 - 6l^2x^2 + l^4\right)} \quad ; \quad \underline{y(x) = \frac{w_o}{120 l EI}\left(x^5 - 2l^2x^3 + l^4 x\right)}$$

At this stage, static can be used to find the remaining reactions. In the FBD,



$$\overset{+}{\uparrow} \Sigma F_y = 0 \Rightarrow \frac{w_o l}{2} - \frac{w_o l}{10} + R_B = 0$$

$$\Rightarrow R_B = -\frac{2}{5}w_o l \Rightarrow \underline{R_B = \frac{2 w_o l}{5}\ (\downarrow)}$$

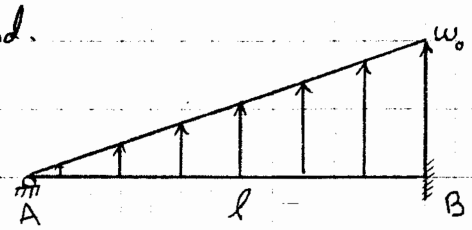
$$\overset{+}{\circlearrowleft} \Sigma M_B = 0 \Rightarrow \frac{w_o l^2}{10} - \frac{w_o l^2}{6} + M_B = 0 \Rightarrow \underline{M_B = \frac{w_o l^2}{15}\ (\circlearrowleft)} \quad (\text{also, } M_B = -M(l))$$

2) Successive integration method:

This method is similar to the double integration procedure except that it starts with the load equation instead of the moment equation. This method is utilized when the loading on the beam is so complicated that it is not easy to obtain the moment equation. Otherwise, double integration method is better. Note that 4 constants, not 2, appear after integrating the load function four times. Thus 4 B.C.s are needed; they include shear & moment B.C.s.

Example 6:

Rework Example 5 utilizing successive integration method.



Solution:

$w(x) = \frac{w_o}{\ell} x$

$V(x) = \int w\,dx = \frac{w_o}{2\ell} x^2 + C_1$

$M(x) = \int V dx = \frac{w_o}{6\ell} x^3 + C_1 x + C_2$ (note that no need for a FBD to obtain $M(x)$)

$EI\theta(x) = \int M dx = \frac{w_o}{24\ell} x^4 + \frac{C_1}{2} x^2 + C_2 x + C_3$

$EIy(x) = \int EI\theta dx = \frac{w_o}{120\ell} x^5 + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$

B.C.s.:

$M(0) = 0$; $y(0) = 0$; $\theta(\ell) = 0$; $y(\ell) = 0$ (4 eqs. & 4 unknowns $\Rightarrow$ ok)

$M(0) = 0 \Rightarrow C_2 = 0$

$y(0) = 0 \Rightarrow C_4 = 0$

$\left.\begin{aligned} \theta(\ell) = 0 &\Rightarrow \frac{w_o}{24}\ell^3 + \frac{C_1}{2}\ell^2 + C_3 = 0 \\ y(\ell) = 0 &\Rightarrow \frac{w_o}{120}\ell^4 + \frac{C_1}{6}\ell^3 + C_3 \ell = 0 \end{aligned}\right\}$ two eqs. & 2 unknowns (C_1 and C_3) $\Rightarrow$ $C_1 = -\frac{w_o \ell}{10}$ and $C_3 = \frac{w_o \ell^3}{120}$

$\Rightarrow$ $\underline{\theta(x) = \frac{w_o}{120\ell EI}(5x^4 - 6\ell^2 x^2 + \ell^4)}$; $\underline{y(x) = \frac{w_o}{120\ell EI}(x^5 - 2\ell^2 x^3 + \ell^4 x)}$

$R_A = V(0) = C_1 \Rightarrow$

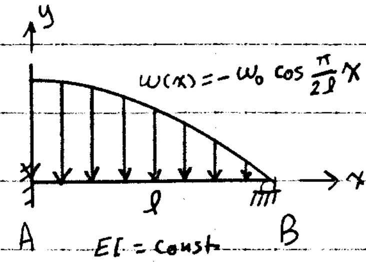
$\underline{R_A = -\frac{w_o \ell}{10} = \frac{w_o \ell}{10} (\downarrow)}$ $\Rightarrow$ from equilibrium, $\underline{R_B = \frac{2w_o \ell}{5} (\downarrow)}$ and $\underline{M_B = \frac{w_o \ell^2}{15} (\circlearrowright)}$

or $R_B = -V(\ell)$
$M_B = -M(\ell)$

Example 7 :

Obtain formulas for the slope and deflection and determine the reactions at A for the beam shown.

Solution :



$w(x) = -w_o \cos \frac{\pi}{2l} x$

$V(x) = \int w\,dx = -w_o \left(\frac{2l}{\pi}\right) \sin \frac{\pi}{2l} x + C_1$

$M(x) = \int V dx = w_o \left(\frac{2l}{\pi}\right)^2 \cos \frac{\pi}{2l} x + C_1 x + C_2$

$EI\theta(x) = \int M dx = w_o \left(\frac{2l}{\pi}\right)^3 \sin \frac{\pi}{2l} x + \frac{C_1}{2} x^2 + C_2 x + C_3$

$EIy(x) = \int EI\theta dx = -w_o \left(\frac{2l}{\pi}\right)^4 \cos \frac{\pi}{2l} x + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$

B. C's. :

$\theta(0) = 0 \Rightarrow C_3 = 0$

$y(0) = 0 \Rightarrow -w_o \left(\frac{2l}{\pi}\right)^4 + C_4 = 0 \Rightarrow C_4 = w_o \left(\frac{2l}{\pi}\right)^4$

$M(l) = 0 \Rightarrow C_1 l + C_2 = 0$ ①

$y(l) = 0 \Rightarrow C_1 \frac{l^3}{6} + C_2 \frac{l^2}{2} + w_o \left(\frac{2l}{\pi}\right)^4 = 0$ ②

Two eqs. and two unks. $\Rightarrow C_1 = \frac{48l}{\pi^4} w_o \; ; \; C_2 = -\frac{48l^2}{\pi^4} w_o \Rightarrow$

$\underline{V(x) = -\left(\frac{2l}{\pi}\right) w_o \sin \frac{\pi}{2l} x + \frac{48l}{\pi^4} w_o}$

$\underline{M(x) = \left(\frac{2l}{\pi}\right)^2 w_o \cos \frac{\pi}{2l} x + \frac{48l}{\pi^4} w_o x - \frac{48l^2}{\pi^4} w_o}$

$\underline{EI\theta(x) = \left(\frac{2l}{\pi}\right)^3 w_o \sin \frac{\pi}{2l} x + \frac{24l}{\pi^4} w_o x^2 - \frac{48l^2}{\pi^4} w_o x}$

$\underline{EIy(x) = -\left(\frac{2l}{\pi}\right)^4 w_o \cos \frac{\pi}{2l} x + \frac{8l}{\pi^4} w_o x^3 - \frac{24l^2}{\pi^4} w_o x^2 + \left(\frac{2l}{\pi}\right)^4 w_o}$

$R_A = V(0) \Rightarrow \underline{R_A = \frac{48l}{\pi^4} w_o \; (\uparrow)}$

$M_A = M(0) \Rightarrow \underline{M_A = \left(\frac{4\pi^2 - 48}{\pi^4}\right) l^2 w_o = -0.0875\, l^2 w_o = 0.0875\, l^2 w_o \; (\circlearrowleft)}$

$R_B = -V(l) \Rightarrow \underline{R_B = \left(\frac{2}{\pi} - \frac{48}{\pi^4}\right) l w_o = 0.144\, l w_o \; (\uparrow)}$

Can you use double integration method in this example?!
Explain!

3) Singularity function method:

The singularity functions permit the expression of $\underline{\underline{any}}$ system of loads as an equivalent distributed load. Thus, $\underline{\underline{one}}$ eq. (for each) w, V, M, θ, and y can be written.

> Note that the quantity inside $\langle \cdots \rangle$ can $\underline{never}$ be $\underline{negative}$.
> $\langle\,\rangle$ = pointed brackets

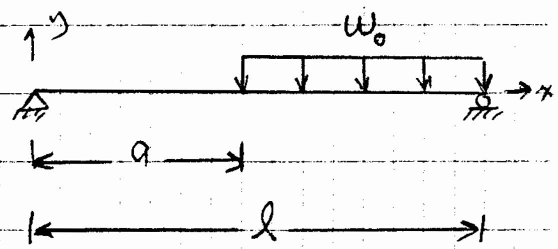
Macaulay functions

$$\langle x-a\rangle^n = \begin{cases} 0 & x<a \\ (x-a)^n & x \geq a \end{cases} \qquad n = 0, 1, 2, 3, \ldots$$

$$\int_0^x \langle \xi - a\rangle^n \, d\xi = \frac{\langle x-a\rangle^{n+1}}{n+1}$$

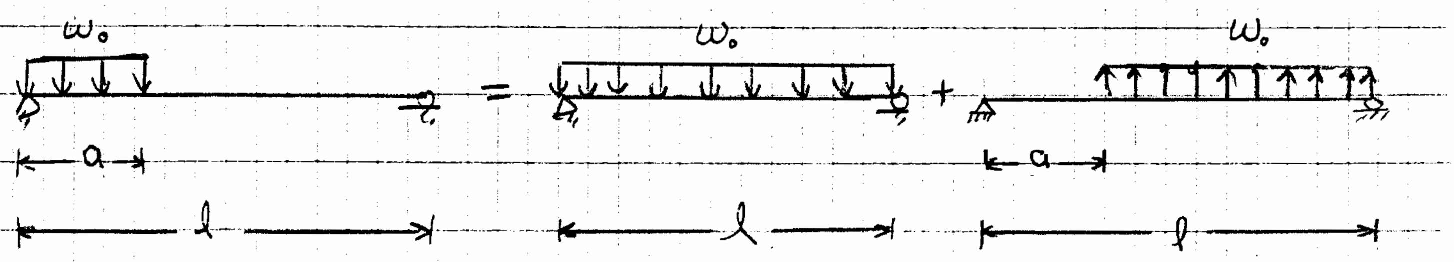
Example 8:

Write the load eqs. using the singularity fun.



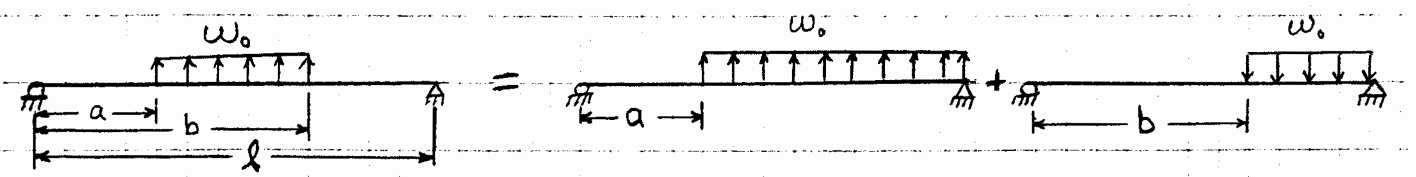
ⓐ $w_e = -w_0 \langle x-a\rangle^0$

ⓑ



$$w_e = -w_0\langle x-0\rangle^0 + w_0\langle x-a\rangle^0$$

ⓒ



$$w_e = w_0\langle x-a\rangle^0 - w_0\langle x-b\rangle^0$$

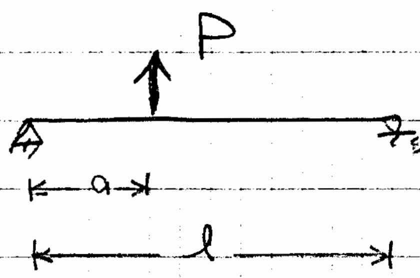
Singularity functions for concentrated force

$$\langle x-a\rangle^{-1} = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

$$\int_0^x \langle \xi - a\rangle^{-1}\, d\xi = \langle x-a\rangle^{0}$$

(Dirac Delta or unit impulse function)

$$w_e = P\langle x-a\rangle^{-1}$$



$$\int_0^x w_e\, d\xi = \int_0^x P\langle \xi - a\rangle^{-1}\, d\xi$$

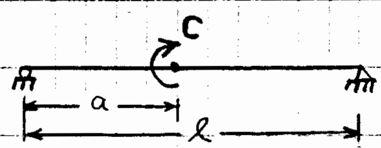
$$= P\langle x-a\rangle^{0} \equiv P$$

Singularity functions for concentrated couple

$$\langle x-a\rangle^{-2} = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

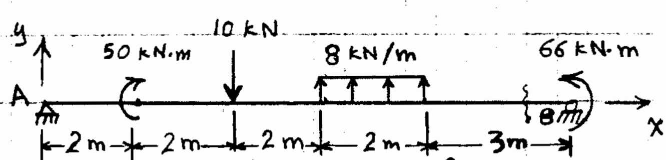
$$\int_0^x \langle \xi - a\rangle^{-2}\, d\xi = \langle x-a\rangle^{-1}$$

$$w_e = C\langle x-a\rangle^{-2} \Longrightarrow$$



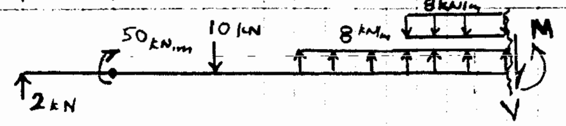
Example 9:

Write a single eq. for M using the sing. fun.



Sol'n: $R_A = 2$ kN ↑ ; $R_B = 8$ kN ↓

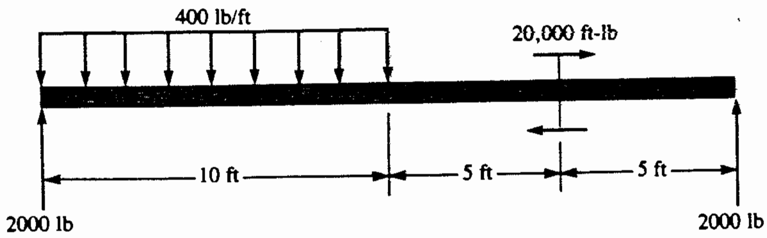
Draw FBD of the last segment ⇒



$$M(x) = 2\langle x-0\rangle^{1} + 50\langle x-2\rangle^{0} - 10\langle x-4\rangle^{1} + 4\langle x-6\rangle^{2} - 4\langle x-8\rangle^{2}$$

EXAMPLE 10

Determine the equivalent distributed load associated with the beam shown in Figure 9–35. Determine the shear and moment equations,- and the slope and deflection equations, using the Macaulay functions and the singularity functions.



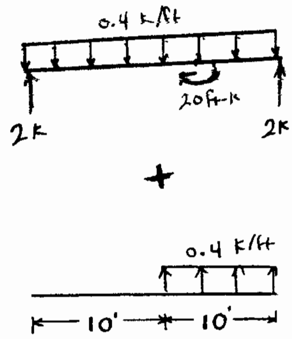


Figure 9–35

SOLUTION The equivalent distributed load corresponding to all applied forces and reactions is

$$\begin{aligned} w_e = {} & 2\langle x-0\rangle^{-1} - [0.4\langle x-0\rangle^{0} - 0.4\langle x-10\rangle^{0}] \\ & + 20\langle x-15\rangle^{-2} + 2\langle x-20\rangle^{-1} \end{aligned} \tag{a}$$

The shearing force equation is obtained by integrating Eq. (a); consequently,

$$\begin{aligned} V(x) = {} & 2\langle x-0\rangle^{0} - [0.4\langle x-0\rangle^{1} - 0.4\langle x-10\rangle^{1}] \\ & + 20\langle x-15\rangle^{-1} + 2\langle x-20\rangle^{0} \end{aligned} \tag{b}$$

The moment equation is obtained by integrating Eq. (b); thus

$$\begin{aligned} M(x) = {} & 2\langle x-0\rangle^{1} - [0.2\langle x-0\rangle^{2} - 0.2\langle x-10\rangle^{2}] \\ & + 20\langle x-15\rangle^{0} + 2\langle x-20\rangle^{1} \end{aligned} \tag{c}$$

Notice that neither equation requires a constant of integration because we included the reactions in the expression for the equivalent distributed load. If the reactions had not been included in w_e, a constant of integration would be required for each integration.

The equations for slope and deflection follow from Eq. (c):

$$\begin{aligned} EIy'(x) = {} & \langle x-0\rangle^{2} - \left[\frac{0.2}{3}\langle x-0\rangle^{3} - \frac{0.2}{3}\langle x-10\rangle^{3}\right] \\ & + 20\langle x-15\rangle^{1} + \langle x-20\rangle^{2} + C_1 \end{aligned} \tag{d}$$

and

$$\begin{aligned} EIy(x) = {} & \tfrac{1}{3}\langle x-0\rangle^{3} - \left[\frac{0.2}{12}\langle x-0\rangle^{4} - \frac{0.2}{12}\langle x-10\rangle^{4}\right] \\ & + 10\langle x-15\rangle^{2} + \tfrac{1}{3}\langle x-20\rangle^{3} + C_1x + C_2 \end{aligned} \tag{e}$$

A constant of integration has been included for each integration that leads to the last two equations. These constants are required so that boundary conditions appropriate to the problem can be satisfied. In the present case, the boundary conditions yield

$$y(0) = 0: \quad C_2 = 0 \tag{f}$$

$$\begin{aligned} y(20) = 0: \quad & \frac{20^3}{3} - \left[\frac{0.2}{12}(20)^4 - \frac{0.2}{12}(10)^4\right] \\ & + 10(5)^2 + 20C_1 = 0 \end{aligned} \tag{g}$$

Accordingly,

$$C_1 = -\frac{500}{24}k \tag{h}$$

Let us write the shear and moment equations for the intervals $0 \le x \le 10$ and $10 \le x \le 15$. From Eqs. (b) and (c), we determine that

$$\left.\begin{array}{lcl} 0 \le x \le 10 & & 10 \le x \le 15 \\ V(x) = 2 - 0.4x & \text{and} & V(x) = 2 - 0.4x + 0.4(x-10) \\ M(x) = 2x - 0.2x^2 & & M(x) = 2x - 0.2x^2 + 0.2(x-10)^2 \end{array}\right\} \tag{i}$$

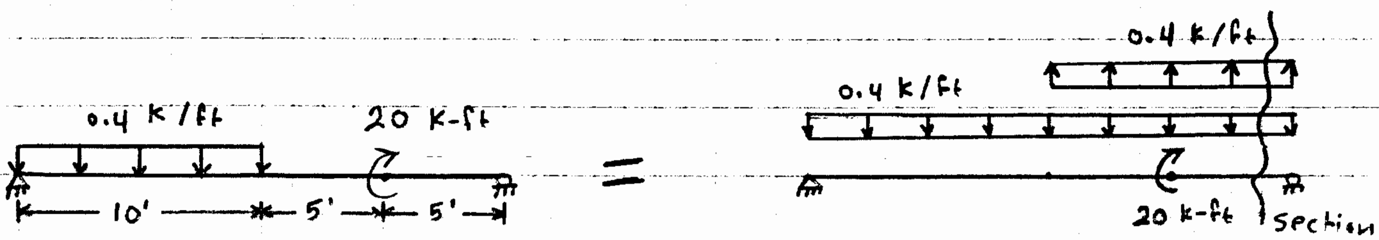
Verify that these equations are correct by drawing appropriate free-body diagrams and invoking force and moment equilibrium as was done in Chapter 6.

Exemple 11

Rework Example 10 (p. 11) by starting with the moment equation.

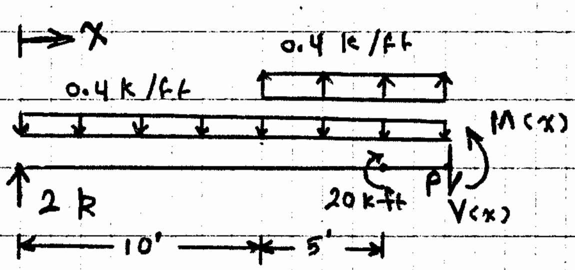
Solution:

Note that once the distributed load starts, it has to continue up to the end of the beam. Thus, the load is redrawn as shown.



Next, make a section (cut) through the last segment of the beam ("near" the right support) after calculating the reactions. Then, draw the FBD of left portion as shown.

note the directions of forces



Now, the moment eq. can be written.

$\Sigma M_P = 0 \Rightarrow$

$$M(x) = 2\langle x-0\rangle^1 - \frac{0.4}{2}\langle x-0\rangle^2 + \frac{0.4}{2}\langle x-10\rangle^2 + 20\langle x-15\rangle^0$$

(Note that the right reaction is not involved in the equation.)

$$EI\,\theta(x) = \langle x-0\rangle^2 - \frac{0.4}{2(3)}\langle x-0\rangle^3 + \frac{0.4}{2(3)}\langle x-10\rangle^3 + 20\langle x-15\rangle^1 + C_1$$

$$EI\,y(x) = \frac{1}{3}\langle x-0\rangle^3 - \frac{0.4}{24}\langle x-0\rangle^4 + \frac{0.4}{24}\langle x-10\rangle^4 + \frac{20}{2}\langle x-15\rangle^2 + C_1 x + C_2$$

B.C.s: $y(0) = 0 \Rightarrow C_2 = 0$ and $y(20) = 0 \Rightarrow C_1 = -125/6 \Rightarrow$

Sing. func. $\left\{ EI\,y(x) = \frac{1}{3}\langle x-0\rangle^3 - \frac{1}{60}\langle x-0\rangle^4 + \frac{1}{60}\langle x-10\rangle^4 + 10\langle x-15\rangle^2 - \frac{125}{6}\langle x-0\rangle \right.$ ⇐ for any value of x

normal functions:

$EI\,y(x) = \frac{1}{3}x^3 - \frac{1}{60}x^4 - \frac{125}{6}x$ ⇐ for $0 \le x \le 10'$

$EI\,y(x) = \frac{1}{3}x^3 - \frac{1}{60}x^4 + \frac{1}{60}(x-10)^4 - \frac{125}{6}x$ ⇐ for $10' \le x \le 15'$

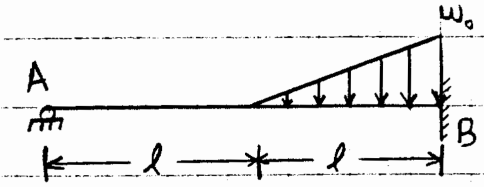
$EI\,y(x) = \frac{1}{3}x^3 - \frac{1}{60}x^4 + \frac{1}{60}(x-10)^4 + 10(x-15)^2 - \frac{125}{6}x$ ⇐ for $15' \le x \le 20'$

These two eqs. can be simplified

Example 12

Given:

The beam shown

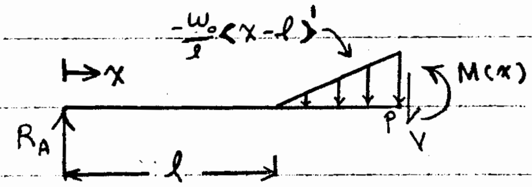


Req'd.:

The reaction at A

Sol'n.:

Since the beam is statically indeterminate, the moment eq. must be expressed in terms of some of the reactions.



$$\Sigma M_p = 0 \Rightarrow M(x) = R_A \langle x-0\rangle^1 - \frac{w_o}{l}\langle x-l\rangle^1\left(\frac{\langle x-l\rangle^1}{2}\right)\left(\frac{\langle x-l\rangle^1}{3}\right)$$

$$= R_A\langle x-0\rangle^1 - \frac{w_o}{6l}\langle x-l\rangle^3$$

$$EI\,\theta(x) = \int M(x)dx = \frac{1}{2}R_A\langle x-0\rangle^2 - \frac{w_o}{24l}\langle x-l\rangle^4 + C_1$$

$$EI\,y(x) = \int EI\,\theta(x)dx = \frac{1}{6}R_A\langle x-0\rangle^3 - \frac{w_o}{120l}\langle x-l\rangle^5 + C_1 x + C_2$$

B.C.s.:

$$\left.\begin{array}{l} y(0)=0 \\ \theta(2l)=0 \\ y(2l)=0 \end{array}\right\} \Rightarrow 3 \text{ eqs. \& } 3 \text{ unknowns } (C_1, C_2, \text{ and } R_A)$$

$$y(0)=0 \Rightarrow C_2 = 0$$

$$\theta(2l)=0 \Rightarrow \frac{1}{2}R_A(2l)^2 - \frac{w_o}{24l}(2l-l)^4 + C_1 = 0 \Rightarrow C_1 = \frac{w_o}{24}l^3 - 2R_A l^2$$

$$y(2l)=0 \Rightarrow \frac{1}{6}R_A(2l)^3 - \frac{w_o}{120l}(2l-l)^5 + C_1(2l) = 0$$

$$\Rightarrow \frac{4}{3}R_A l^3 - \frac{w_o}{120}l^4 + \frac{w_o}{12}l^4 - 4R_A l^3 = 0 \Rightarrow \boxed{R_A = \frac{9}{320}w_o l}$$