#20

Transformation of Stress

Theory
&
Examples

- **Triaxial** states of stress are shown in Fig. ①.
  Only positive stresses on positive faces are shown

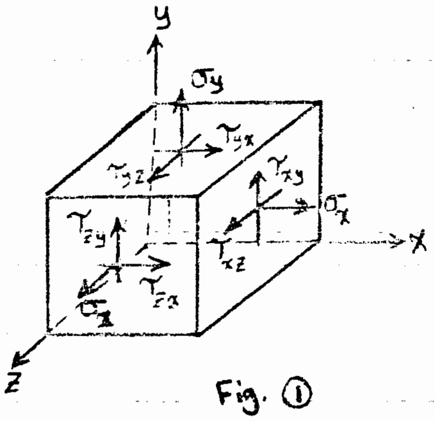


Fig. ①

- **Biaxial** states of stress (Plane Stress):
  When all stresses act in the same plane.
  ⇒ Work in 2-D as shown in Fig. ② in the x-y directions.

⊛ ⊕ Stresses are shown.

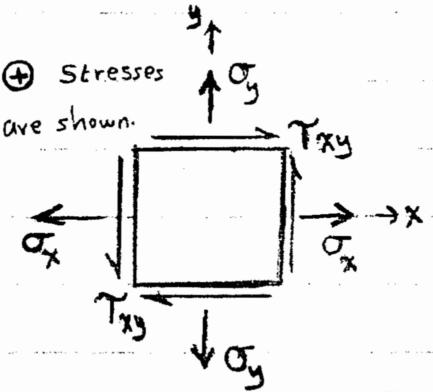


Fig. ②

It is possible to have the orientation in different directions, as shown in Fig. ③ In the ξ-η (xi-eta) directions

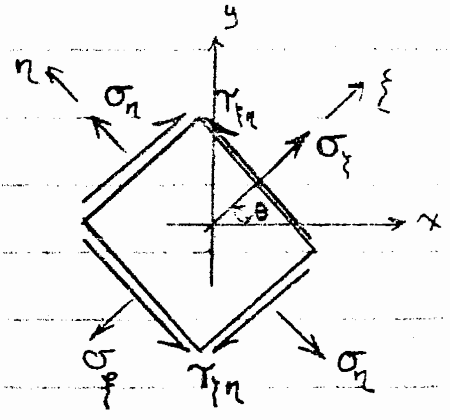


Fig. ③

Now, relationships between the stresses in the x-y and ξ-η directions are sought.

From Fig. ④, the sum of forces in the ξ and η directions must be zero.

* Note that forces not stresses are added..

(2 eqs. & 2 unkns.)

$\sum F_\xi = 0$ & $\sum F_\eta = 0 \Rightarrow$

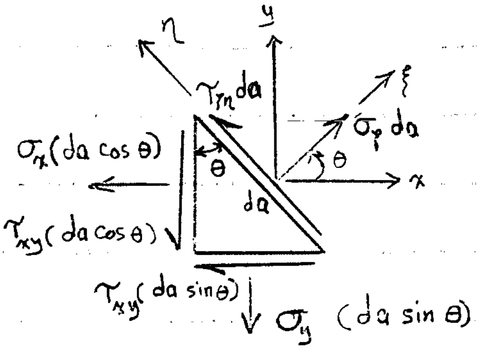


Fig. ④

$$\sigma_\xi = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{\xi\eta} = -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$

Recall that

$$\cos^2\theta = (1 + \cos 2\theta)/2$$

$$\sin^2\theta = (1 - \cos 2\theta)/2$$

$$\sin\theta \cos\theta = \sin 2\theta / 2$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$\Rightarrow$

$$\sigma_\xi = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \boxed{1}$$

$$\tau_{\xi\eta} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad \boxed{2}$$

Similarly,

$$\sigma_\eta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \qquad \boxed{3}$$

$$\tau_{\eta\xi} = \tau_{\xi\eta} \qquad \boxed{4}$$

* Note that $\sigma_x + \sigma_y = \sigma_\xi + \sigma_\eta = I$ (Invariant with any θ) $\Leftarrow$ Use it to check.

When σ & τ on any two orthogonal faces are known, the stress compontents on all (any) faces (plane stress) can be calculated.

Principal Normal Stresses :

$\sigma_\xi = f(\theta) \Rightarrow$ to get $\sigma_{\xi_{max}}$, set $\frac{d\sigma_\xi}{d\theta} = 0 \Rightarrow$ find $\theta_p \Rightarrow \sigma_{\xi_{max}}$

$$\frac{d\sigma_\xi}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta_p + 2\tau_{xy}\cos 2\theta_p = 0$$

Dividing by $\cos 2\theta$, $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$ $\boxed{5}$

From the equation above, the figure shown can be constructed.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \boxed{6}$$

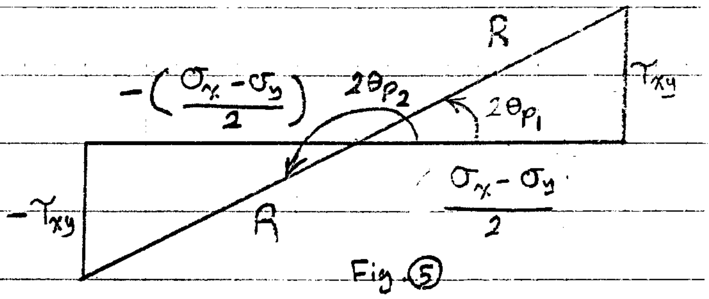


Fig ⑤

$$\sin 2\theta_{p_1} = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta_{p_1} = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\sin 2\theta_{p_2} = \frac{-\tau_{xy}}{R} \quad ; \quad \cos 2\theta_{p_2} = \frac{-(\sigma_x - \sigma_y)/2}{R}$$

$\boxed{7}$

$$\Rightarrow \boxed{\sigma_{\substack{max \\ min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_x + \sigma_y}{2} \pm R} \Leftarrow \text{Principal Normal Stresses}$$

The directions are given by θ_{p_1} and θ_{p_2}

Note that $2\theta_{p_2} = 2\theta_{p_1} + \pi \Longrightarrow \theta_{p_1} \perp \theta_{p_2}$

Also note that $\underline{\underline{\tau_{\xi\eta} = 0}}$ on the planes which the principal normal stresses act.

Principal Shearing Stresses :

⟸ usually called maximum (not principal) τ

$\tau_{\xi\eta} = f(\theta) \Rightarrow$ The value of θ_s can be obtained by setting $\frac{d\tau_{\xi\eta}}{d\theta} = 0$.

$$\Rightarrow \tan 2\theta_s = \frac{-\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}} \qquad \boxed{8}$$

There are 2 possible values for θ_s

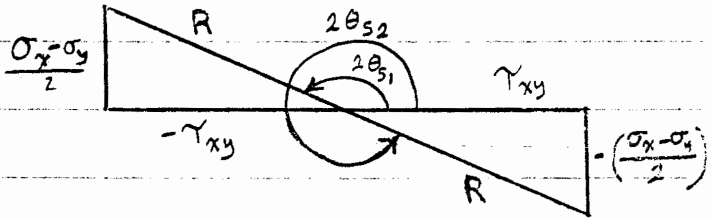


Fig. ⑤

$$\Rightarrow \sin 2\theta_{s_1} = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\cos 2\theta_{s_1} = \frac{-\tau_{xy}}{R}$$

$$\sin 2\theta_{s_2} = \frac{-(\sigma_x - \sigma_y)/2}{R} \quad ; \quad \cos 2\theta_{s_2} = \frac{\tau_{xy}}{R}$$

$$\Rightarrow \boxed{\tau_{\substack{max \\ min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm R}$$

⟵ Principal Shearing Stresses $\boxed{9}$

$$\boxed{\sigma_\xi = \sigma_\eta = \frac{\sigma_x + \sigma_y}{2}}$$

⟵ Normal Stresses on the Planes of Principal Shearing Stresses $\boxed{10}$

Note that $2\theta_{s_2} = 2\theta_{s_1} + \pi \Rightarrow \theta_{s_1} \perp \theta_{s_2}$

Also note that $\tan 2\theta_p$ is the negative reciprocal of $\tan 2\theta_s$:

$\tan 2\theta_p = -1/\tan 2\theta_s$

$\Rightarrow 2\theta_s = 2\theta_p + \pi/2 \Rightarrow \boxed{\theta_s = \theta_p + 45^\circ}$ $\boxed{11}$

Thus, there is a 45° angle between the planes of principal normal and shearing stresses.

Mohr's Circle :

The general equation of the circle is

$$(x-a)^2 + (y-b)^2 = R^2 \qquad \boxed{12}$$

By rearranging eq. $\boxed{1}$ and squaring both sides of eqs. $\boxed{1}$ & $\boxed{2}$, and then adding, the following equation is obtained :

$$\left(\sigma_\xi - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{\xi\eta}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad \boxed{13}$$

By comparing eq. $\boxed{13}$ with eq. $\boxed{12}$, it can be seen that

$x = \sigma_\xi$

$a = (\sigma_x + \sigma_y)/2 = \sigma_{average}$

$y = \tau_{\xi\eta}$

$b = 0$

$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$

Thus, by constructing a circle with the properties above, and by refering to Fig. ⑤, the values of σ_ξ, σ_η, and $\tau_{\xi\eta}$ with any θ can be found; this includes the principal stresses and their directions. This circle is called Mohr's Circle because Mohr brought the idea of such a circle. It has several applications, other than stresses.

* Steps for constructing Mohr's circle and determining the principal stresses & directions and σ & τ with any θ :

① Draw the σ-τ axes in the H-V directions with "appropriate" scale. } H = Horizontal, V = Vertical

② Put the points x (σ_x, τ_{xy}) and y $(\sigma_y, -\tau_{xy})$ on the figure.

③ Connect the two points x and y by a straight line. The point of intersection of the line xy and the σ-axis is the center of the circle C, and CX & Cy are two radii of such a circle.

④ Construct the circle with C as its center and CX (or Cy) the radius.

⑤ The points of intersection of the circle and the σ-axis are the principal normal stresses; the one to the right is the maximum, and the one to the left is the minimum.

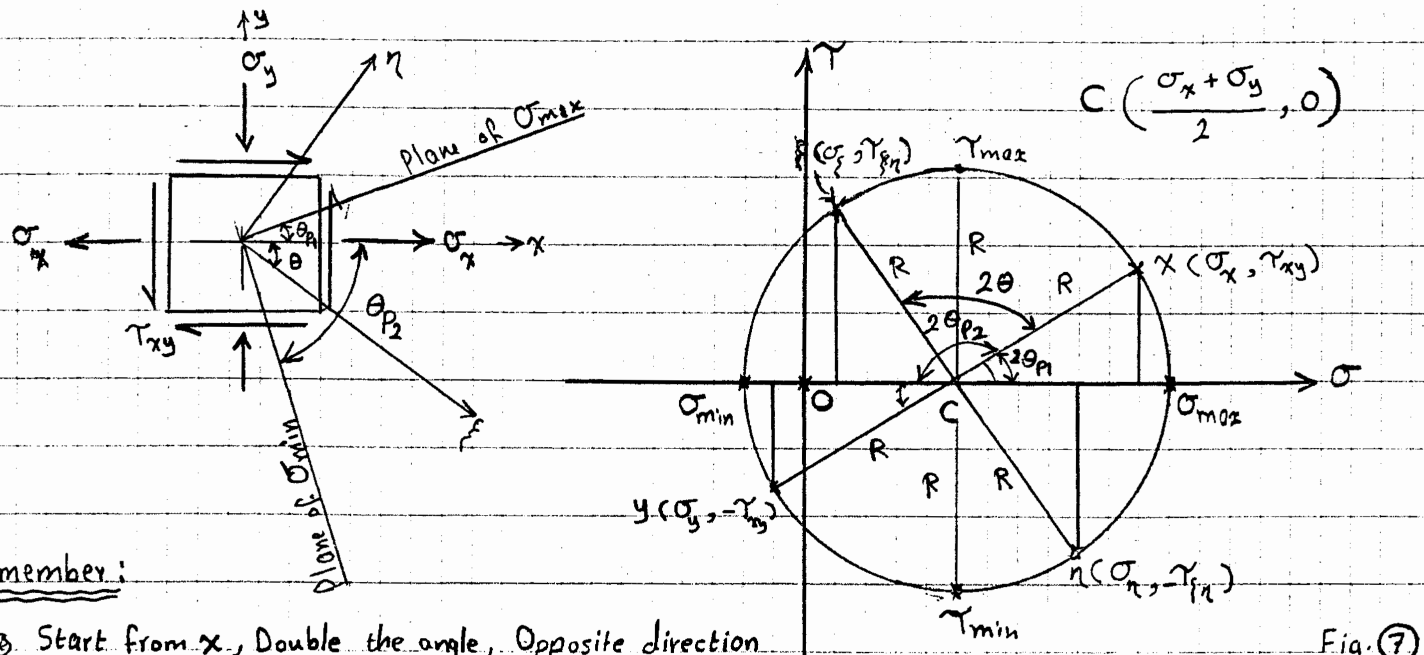
⑥ The radius of the circle is τ_{max} and $\tau_{min} = -R$.

⑦ The angle measured from the x-axis to σ_{max} gives $-2\theta_{p_{1,2}}$ (i.e., double the angle in the opposite direction), and the angle measured from X to σ_{min} is $-2\theta_{p_{2,1}}$.

⑧ The angle measured from X to τ_{max} is $-2\theta_{s_{1,2}}$, and the angle from X to τ_{min} is $-2\theta_{s_{2,1}}$

⑨ The similar triangles show in Fig. ⑦ are used to calculate the required values (stresses and their directions).

⑩ To calculate the stresses on planes oriented $\theta°$ from the x-axis on the real plane, go 2θ from the x-axis in the opposite direction on the imaginary plane (Mohr's circle), and then draw a straight line passing through C. Use the triagles shown in Fig. ⑦ to determine the stress values.



Remember:

⊗ Start from x, Double the angle, Opposite direction

Fig. ⑦

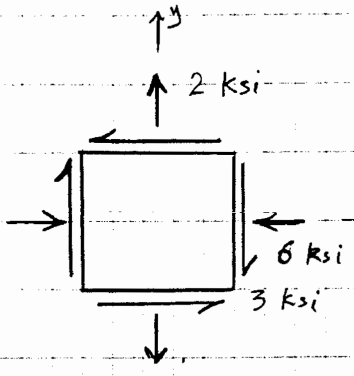
• You can go in the same direction (not opposite) if one of the following is done:

① Reverse the "sign convention" for shear $[\ \uparrow\boxed{+}\downarrow\ ]$

② Plot $X(\sigma_x, -\tau_{xy})$, $y(\sigma_y, \tau_{xy})$

③ Plot $\sigma-\tau$ axes as $\longrightarrow \sigma$

Example 1:



Given:

The state of stress shown

Req.d.:

a) The principal stresses & directions

b) σ & τ associated with an element oriented 10° cw of the element shown.

Show the results on properly oriented elements.

Use the equations for the solution

Solution:

a) $$\sigma_{\substack{max\\min}} = \frac{\sigma_x+\sigma_y}{2} \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = \frac{-6+2}{2} \pm\sqrt{\left(\frac{-6-2}{2}\right)^2+(-3)^2} = -2 \pm 5$$

$$\Rightarrow \quad \underline{\underline{\sigma_{max} = 3 \text{ ksi}}} \quad ; \quad \underline{\underline{\sigma_{min} = -7 \text{ ksi}}}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = \frac{2(-3)}{-6-2} = 0.75 \Rightarrow 2\theta_{p_1} = 36.87^\circ \quad , \quad 2\theta_{p_2} = 216.87^\circ \ (-143.13^\circ)$$

$$\Rightarrow \quad \underline{\underline{\theta_{p_1} = 18.43^\circ}} \quad ; \quad \underline{\underline{\theta_{p_2} = 108.43^\circ \ (-71.57^\circ)}}$$

To see θ_{p_1} corresponds to σ_{max} or σ_{min}, substitute θ in Eq. $\boxed{1}$ by $\theta_{p_1} = 18.43^\circ \Rightarrow$

$\Rightarrow \sigma_\xi = \frac{-6+2}{2} + \frac{-6-2}{2} \cos 36.87^\circ - 3 \sin 36.57^\circ = -7 \text{ ksi} \Rightarrow \theta_{p_1}$ is the direction of σ_{min} as shown below.

$\tau_{\substack{max\\min}} = \pm R \Rightarrow \underline{\underline{\tau_{max} = 5 \text{ ksi}}}$; $\underline{\underline{\tau_{min} = -5 \text{ ksi}}}$ $\quad \underline{\underline{\sigma_\xi = \sigma_\eta = \frac{-6+2}{2} = -2 \text{ ksi}}}$

$\tan 2\theta_s = -1/0.75 \Rightarrow \underline{\underline{\theta_{s_1} = -26.57^\circ}}$; $\underline{\underline{\theta_{s_2} = 63.43^\circ}}$ $\quad \tau(-26.57^\circ) = -5 \text{ ksi} = \tau_{min} \Leftrightarrow \theta_{s_1}$

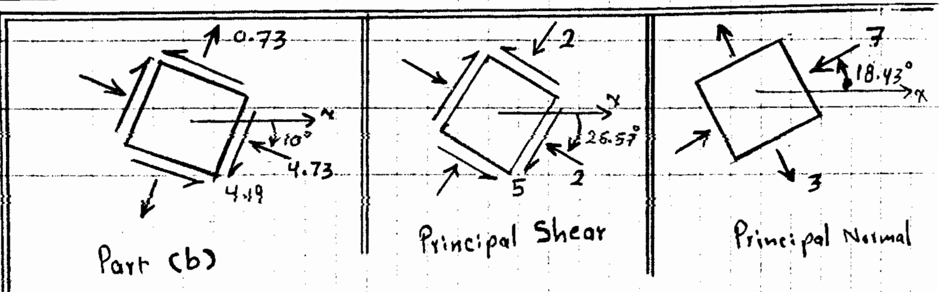
b) From Eqs. $\boxed{1}$ to $\boxed{4}$

$\underline{\underline{\sigma_\xi(-10^\circ) = -4.73 \text{ ksi}}}$

$\underline{\underline{\sigma_\eta(-10^\circ) = 0.73 \text{ ksi}}}$

$\underline{\underline{\tau_{\xi\eta}(-10^\circ) = \tau_{\eta\xi}(-10^\circ) = -4.19 \text{ ksi} \quad (\pm)}}$

Note that $\Sigma\sigma_i = -4$ (always)



Part (b) Principal Shear Principal Normal

All units are in ksi

Example 2 :

Rework Example 1 using Mohr's circle.

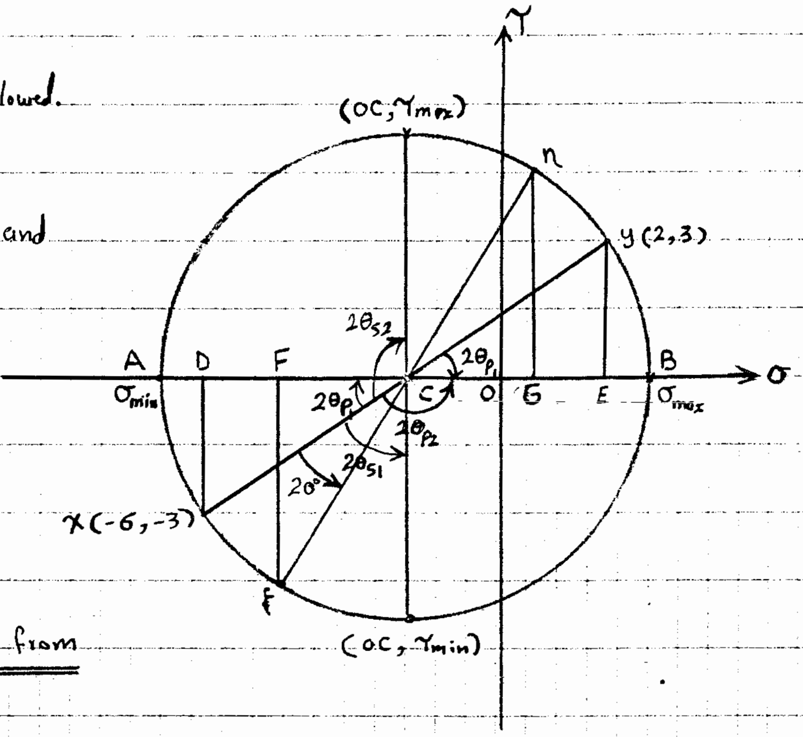
Solution :

The steps given on pp. 5, 6 will be followed.

$OC = \frac{OE + OD}{2} = \frac{2-6}{2} = -2$

Using the triangle CxD or CyE, R and $2\theta_p$ can be calculated. ⇒

$\tan 2\theta_p = \frac{xD}{CD} = \frac{|xD|}{|OD| - |OC|}$



$= \frac{3}{6-2} = 0.75$

$\Rightarrow 2\theta_{p_1} = 36.87° \Rightarrow$

$\Rightarrow$ $\underline{\underline{\theta_{p_1} = 18.43° \text{ CCW } (\circlearrowleft) \text{ measured from the x-axis to the axis of } \sigma_{min}}}$

* Remember : Double the angle in opposite direction

$\Rightarrow 2\theta_{p_2} = 180° - 2\theta_{p_1}$

$\Rightarrow$ $\underline{\underline{\theta_{p_2} = 71.57° \text{ CW } (\circlearrowright) \text{ measured from x to } \sigma_{max}}}$

$R = Cx = Cy = yE/\sin 2\theta_{p_1} = 3/\sin 36.87° = 5$

$\sigma_{max} = OB = CB - |OC| = R - |OC| = 5 - 2 \Rightarrow \underline{\underline{\sigma_{max} = 3 \text{ ksi}}}$

$\sigma_{min} = \underset{\Uparrow}{-}|OA| = -(|OC| + |CA|) = -(|OC| + R) = -(2+5) \Rightarrow \underline{\underline{\sigma_{min} = -7 \text{ ksi}}}$

Take care of the signs by inspection.

$\tau_{\underset{min}{max}} = \pm R \Rightarrow \underline{\underline{\tau_{max} = 5 \text{ ksi}}} \; ; \; \underline{\underline{\tau_{min} = -5 \text{ ksi}}}$

$2\theta_{s_1} = 90° - 2\theta_{p_1} = 90 - 36.87° \Rightarrow \underline{\underline{\theta_{s_1} = 26.57° \text{ CW } (\circlearrowright) \text{ measured from x to } \tau_{min}}}$

$2\theta_{s_2} = 90° + 2\theta_{p_1} = 90 + 36.87° \Rightarrow \underline{\underline{\theta_{s_2} = 63.43° \text{ CCW } (\circlearrowleft) \text{ measured from x to } \tau_{max}}}$

b) $\sigma_f(-10°) = OF = -(|OC| + |CE|) = -[|OC| + R\cos(2\theta_{p_1} + 20°)] = -[2 + 5\cos(56.87°)] \Rightarrow \underline{\underline{\sigma_f(-10) = -4.73 \text{ ksi}}}$

$\sigma_n(-10°) = OG = R\cos(2\theta_{p_1} + 20°) - |OC| = 5\cos(56.87) - 2 \Rightarrow \underline{\underline{\sigma_n(-10°) = 0.73 \text{ ksi}}}$

$\tau_{\underset{nf}{fn}} = \pm \underbrace{R \sin(56.87°)}_{fF = nG} \Rightarrow \underline{\underline{\tau_{\underset{nf}{fn}}(-10°) = \pm 4.19 \text{ ksi}}}$

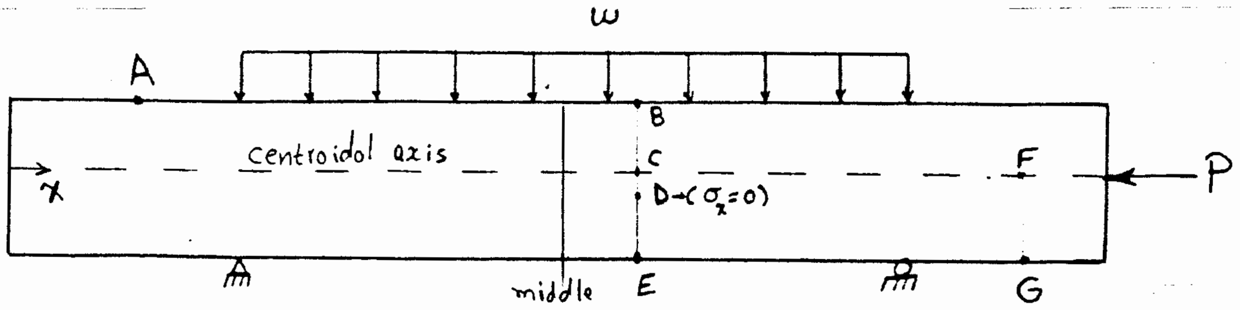
Example 3:

Given :

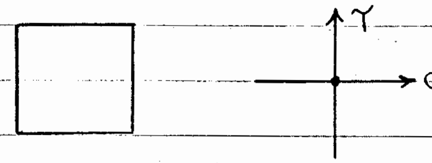
The beam shown

Req'd. :

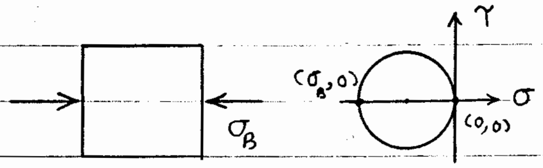
Qualitatively, sketch Mohr's circle for the points A to G.



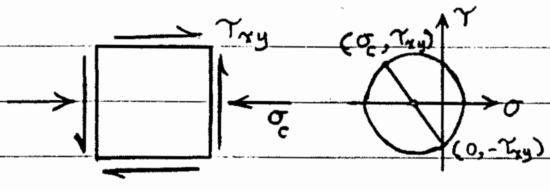
Solution :



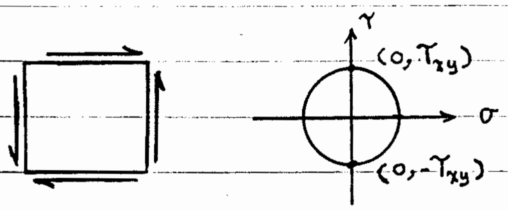
A



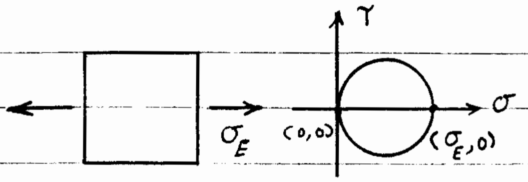
B "$\underline{\underline{\omega}}$ ignored"



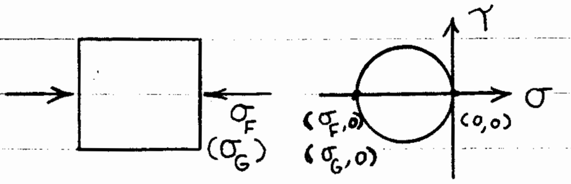
C



D



E



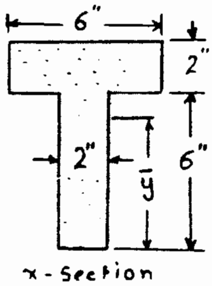
F & G

* **Note** that σ_y is always assumed zero (ignored) in beams

Example 4:

Given:

The beam shown



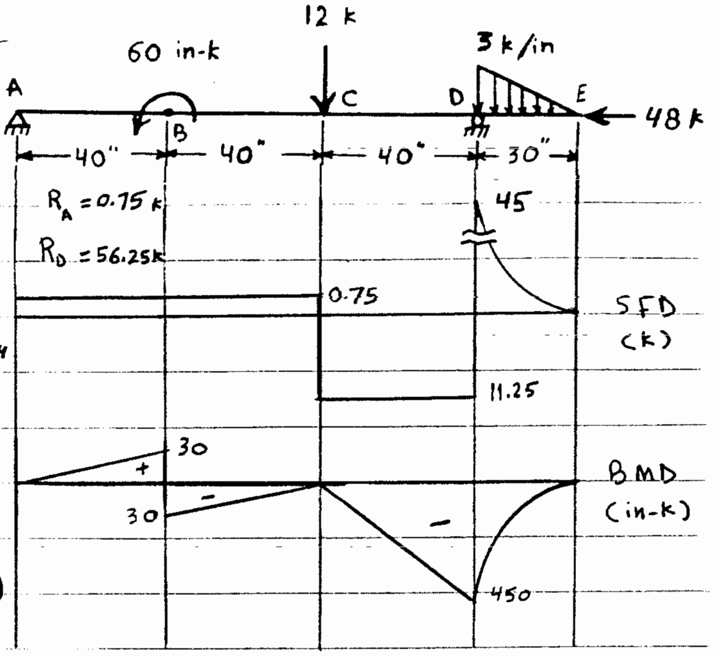
Req'd.:

The principal normal and shearing stresses and their directions at the point(s) of maximum stresses

$\bar{y} = 5"$

$I = 136 \text{ in}^4$

$A = 24 \text{ in}^2$



Soln.:

$M_{max} = 450$ in-k $\overset{T}{\underset{C}{\frown}}$ @ D

This M will give σ_{max} (both T & C)

$V_{max} = 45$ k @ D also

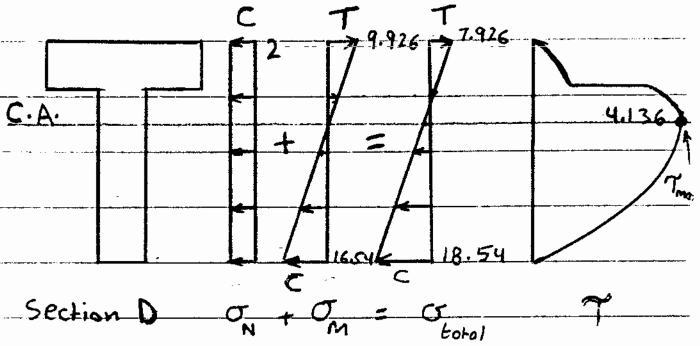
$$\sigma = \sigma_N + \sigma_M = \pm\frac{N}{A} \pm \frac{My}{I}$$

$$\sigma_{top} = \frac{-48}{24} + \frac{450(3)}{136} = -2 + 9.926$$

$$= 7.926 \text{ ksi (T)}$$

$$\sigma_{bottom} = -\frac{48}{24} - \frac{450(5)}{136} = -2 - 16.54$$

$$= 18.54 \text{ ksi (C)}$$



$$\sigma_{C.A.} = -\frac{48}{24} + 0 = 2 \text{ ksi (C)}$$

$$\tau_{top} = \tau_{bottom} = 0$$

$$\tau_{C.A.} = \tau_{max} = \frac{VQ}{Ib} = 45(5\times2\times2.5)/136(2) = 4.136 \text{ ksi}$$

σs above are all σ_x

$\sigma_y = 0$ at all points (always the case in beam theory)

We need to calculate the principal stresses at 3 points (top, bottom, N.A. of section D) ⇒ choose the maximum normal (T & C) and shearing stresses.

1) top of D: (Mohr's circle is used for the stress values below, not shown)

$\sigma_x = 7.926$ ksi , $\sigma_y = 0$, $\tau_{xy} = 0$

⇒ $\sigma_{max} = 7.926$ ksi , $\sigma_{min} = 0$, $\tau_{max} = 3.963$ ksi , $\tau_{min} = -3.963$ ksi

2) bottom of D: $\sigma_x = -18.54$ ksi , $\sigma_y = 0$, $\tau_{xy} = 0$

⇒ $\sigma_{max} = 0$, $\sigma_{min} = -18.54$ ksi , $\tau_{max} = 9.27$ ksi , $\tau_{min} = -9.27$ ksi

3) Centroidal Axis: $\sigma_x = -2$ ksi , $\sigma_y = 0$, $\tau_{xy} = 4.136$ ksi

⇒ $\sigma_{max} \simeq 3.2$ ksi , $\sigma_{min} \simeq -5.2$ ksi , $\tau_{max} \simeq 4.2$ ksi , $\tau_{min} \simeq -4.2$ ksi

⇒ $\sigma^T_{max} = 7.926$ ksi , $\sigma^C_{max} = 18.54$ ksi , $\tau_{max} = 9.27$ ksi @ D

⊗ **Important note:** When τ_{xy} is zero at a point, then σ_x and σ_y are themselves principal stresses at that particular point (as seen above).