Deflection of Beams

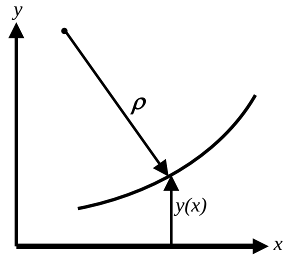
Theory & Examples

* *Moment-Curvature Relation (developed earlier):*

$$\frac{1}{\rho} = \frac{M}{EI}$$

From calculus, the curvature of the plane curve shown is given by

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

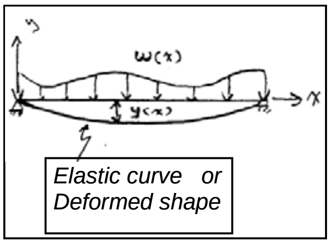


For "very small" deformation (as it is the case in most engineering problems), $(dy/dx)^2 << 1$
Thus,

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{M}{EI} = \frac{d^2y}{dx^2} \quad \Leftarrow \quad \textit{\textbf{y is the deflection}}$$

$$\Rightarrow \quad M = EI\,\frac{d^2y}{dx^2}$$



Recall that $V(x) = \frac{dM}{dx}$ & $w(x) = \frac{dV}{dx}$

Thus, the summary is

$$\boxed{w(x) = \frac{d^2M}{dx^2} = EI\,\frac{d^4y}{dx^4} = load} \qquad (1)$$

$$\boxed{V(x) = \frac{dM}{dx} = EI\,\frac{d^3y}{dx^3} = shear} \qquad (2)$$

$$\boxed{M(x) = EI\,\frac{d^2y}{dx^2} = Moment} \tag{3}$$

$$\boxed{\theta(x) = \frac{dy}{dx} = slope} \tag{4}$$

$\Rightarrow$

$$\boxed{V = \int w\,dx}$$

$$\boxed{M = \int V\,dx = \int\int w\,dx\,dx}$$

$$\boxed{\theta = \int \frac{M}{EI}\,dx = \int\int\int \frac{w}{EI}\,dx\,dx\,dx}$$

$$\boxed{y = \int \theta\,dx = \int\int\int\int \frac{w}{EI}\,dx\,dx\,dx\,dx}$$

The deflection of the beam is needed for two main reasons:

1) **To limit the maximum deflection (i.e. $y_{max} \leq y_{allowable}$)**
2) **To determine the reactions in statically indeterminate (SI) problems**

If the beam is ***designed based on the maximum allowable deflection***, this is called **"design for stiffness"**. If the ***design is based on limiting the maximum (allowable) stress***, it is called **"design for strength"**. In most applications, the **stress controls** (i.e. ***limiting the stress is more important than limiting the deflection because deflections are usually "very small" in "typical" structures***). Thus, the second reason above (**SI problems)** is more important than the first one (**to limit the maximum deflection**).

There are many methods for calculating slopes and deflections of beams. In this course, only three methods are covered. In CE 305 (Structural Analysis I), several methods, including energy and computer procedures, are discussed in details.

The three methods are

1) **Double Integration**
2) **Successive Integration**
3) **Singularity function**

In fact, these three methods have the same theoretical basis; thus, they could be considered as one way, with different branches, for determining deflections. It is the elementary, fundamental, or basic method of integration.

The **deflection due to the moment** **only** will be discussed here. The deflection due to the shear is discussed in CE 305 and other courses. However, y_V ***is usually much less than*** y_M***. Therefore,*** y_V ***is negligible in most cases.***

1) Double integration method:

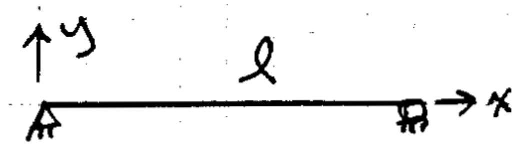
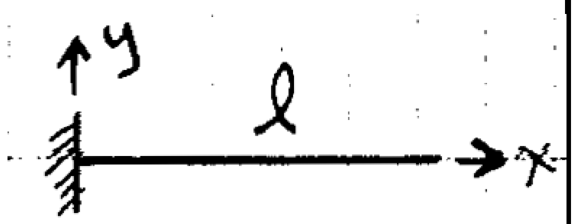
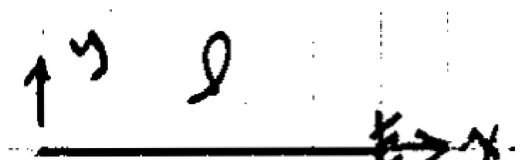
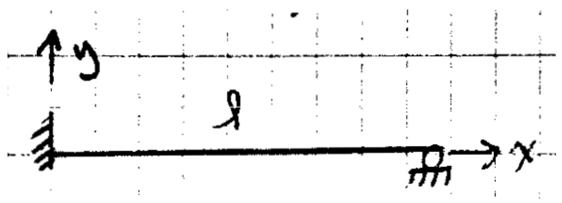
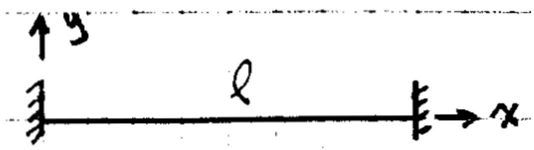
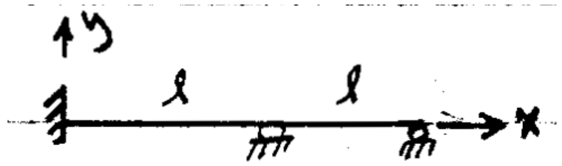
If the moment equation is known or it can be obtained easily, then by **integrating twice (double), the deflection equation can be determined**. In this case, ***two integration constants*** ***for*** ***each*** ***moment equation appear***; therefore, ***two boundary conditions*** **(B.C's.)** for each equation are needed. *Note that there could be more than one moment equation in a beam, depending on the loading conditions*.

In **statically indeterminate** beams, the moment equation can not be written explicitly, but it must be written in terms of some of the unknown reactions. Thus, more than two boundary conditions are needed in order to solve for the two constants and the unknown reactions, as will be seen in the examples. *[These **"extra" reactions*** *are usually called* ***redundants****.]* In general, ***the number of B. C.'s has to equal to 2 plus the degree of statically indeterminacy*** of the beam (***n***), or

B.C.'s = 2 + n

Example 1:

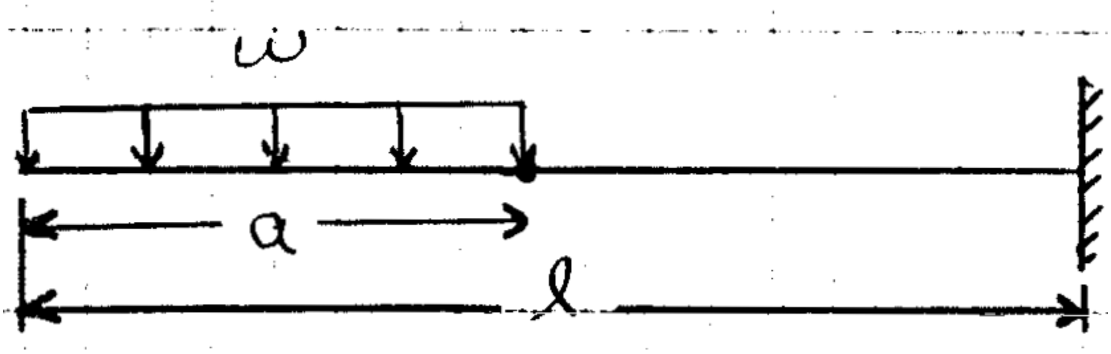
Write the B.C.'s for each of the beams shown below [1(a) to 1 (f)].

 $y(0) = 0$ $y(\ell) = 0$ **1(a)**	 $y(0) = 0$ $\theta(0) = 0$ **1(b)**
 $y(\ell) = 0$ $\theta(\ell) = 0$ **1(c)**	 $y(0) = 0$ $y(\ell) = 0$ $\theta(0) = 0$ **1(d)**
 $y(0) = 0$ $y(\ell) = 0$ $\theta(0) = 0$ $\theta(\ell) = 0$ **1(e)**	 $y(0) = 0$ $y(\ell) = 0$ $y(2\ell) = 0$ $\theta(0) = 0$ **1(f)**

The discussion above about B.C.'s is true for beams with a single moment equation. If the beam has more than one moment equation, then the total number of constants is equal to 2 times the number of equations. Thus, two B.C.'s are not enough to solve for all the constants. Therefore, *the concept of **continuity conditions*** (**C.C.'s**) *is introduced*. That is, ***the slope and deflection must be continuous between adjacent intervals***. These continuity conditions give additional or supplementary equations which make it possible to solve for all the constants, as illustrated below. However, as the number of moment equations increases the number of unknown constants increases as well, giving a large number of equations which have to be solved simultaneously. This could be very tedious and time-consuming; thus, this method becomes impractical, and a better one, called ***singularity function*** method, is introduced, as will be discussed later. Because of that, beams with one moment equation only are covered by this method as well as by the method of successive integration.

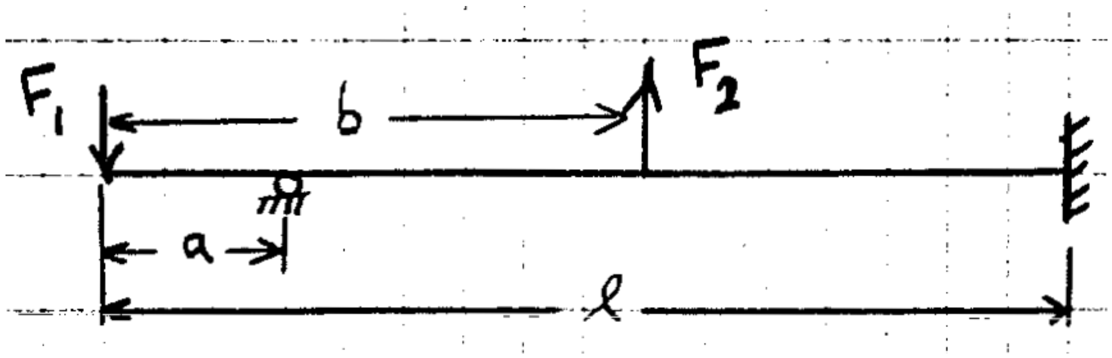
Example 2:

Write the B.C.'s and the C.C.'s for each of the beams shown below [2(a) & 2(b)].



$y(a-) = y(a+)$	**$y(\ell) = 0$**
$\theta(a-) = \theta(a+)$	**$\theta(\ell) = 0$**
$\Uparrow$	$\Uparrow$
C.C.'s	**B.C.'s**

2(a)

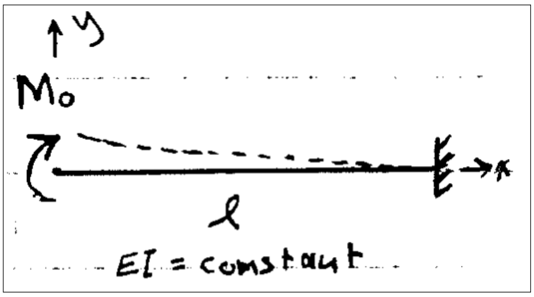


y(a) = 0	y(b-) = y(b+)	y(ℓ) = 0
θ(a-) = θ(a+)	θ(b-) = θ(b+)	θ(ℓ) = 0
⇑	⇑	⇑
B.C.'s & C.C.'s	**C.C.'s**	**B.C.'s**

2(b)

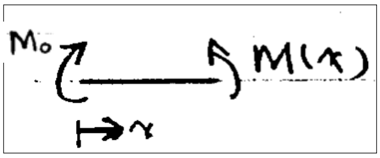
Example 3:

Derive an expression for the elastic curve (deflection) and find the maximum y & θ in the beam shown.



Solution:

From the ***FBD*** shown, the moment equation can be written:



$M(x) = M_o$

$$EI\ \theta(x) = \int M(x)dx = \int M_o\, dx = M_o\, x + C_1$$

$$EI\ y(x) = \int EI\,\theta(x)dx \ = \int (M_o\, x + C_1)dx = \frac{1}{2} M_o\, x^2 + C_1 x + C_2$$

B.C.'s : y(ℓ) = 0 ; θ(ℓ) = 0 [two B.C.'s and two constants ⇒ OK]

$\theta(\ell) = 0 \Rightarrow M_o\,\ell + C_1 = 0 \Rightarrow C_1 = -M_o\,\ell$

$y(\ell) = 0 \Rightarrow \frac{1}{2} M_o\,\ell^2 + (-M_o\ell)\ell + C_2 = 0 \Rightarrow C_2 = \frac{1}{2} M_o\,\ell^2$

$$\Rightarrow \qquad \boxed{EI\,\theta(x) = M_o\,(x - \ell)}$$

$$\boxed{EI\;y(x) = M_o\left(\frac{x^2}{2} + \ell x + \frac{\ell^2}{2}\right)}$$

θ_{max} & y_{max} are at the free end. ***In general,*** ***<u>y_{max}</u> is <u>always</u> at the free end or at the point Where*** $\theta = \dfrac{dy}{dx} = 0$.

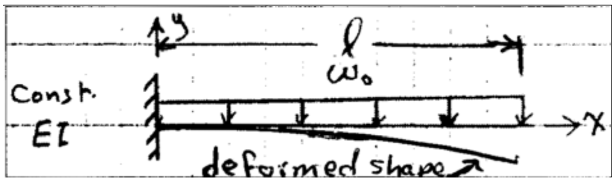
$\theta_{max} = \theta(0) \quad \Rightarrow \quad \theta_{max} = -M_o\ell = M_o\ell$ (*cw*) @ <u>x = 0</u>
$y_{max} = y(0) \quad \Rightarrow \quad y_{max} = M_o\ell^2/2$ (↑) @ <u>x = 0</u>

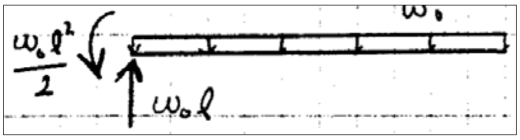
<u>Example 4</u>:

Derive equations for θ and y for the beam shown.

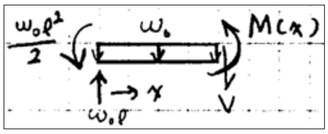
<u>Solution</u>:

$M(x) = w_o\ell\, x - \frac{1}{2}\, w_o\ell^2 - w_o\, x^2/2$





FBD's



$$EI\theta = \int M\,dx = \int (w_o\ell x - \frac{1}{2}w_o\ell^2 - w_o\,\frac{x^2}{2}\,)\;dx$$

$$= \frac{1}{2}w_o\,\ell x^2 - \frac{1}{2}w_o\ell^2 x - \frac{1}{6}w_o\,x^3 + C_1 = \frac{1}{6}w_o x^3 + \frac{1}{2}w_o\ell x^2 - \frac{1}{2}w_o\ell^2 x + C_1$$

$$EIy = \int EI\theta dx = \int\left(-\frac{1}{6}w_o x^3 + \frac{1}{2}w_o\ell x^2 - \frac{1}{2}w_o\ell^2 x + C_1\right)$$

$$= -\frac{1}{24}w_o x^4 + \frac{1}{6}w_o\ell x^3 - \frac{1}{4}w_o\ell^2 x^2 + C_1 x + C_2$$

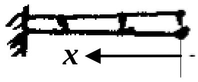
B.C.'s: $\quad \theta(0) = 0 \quad \Rightarrow \quad C_1 = 0$
$\qquad\qquad y(0) = 0 \quad \Rightarrow \quad C_2 = 0 \quad \Rightarrow$

$$\boxed{\theta(x) = \frac{w_o}{6EI}(-x^3 + 3\ell x^2 - 3\ell^2 x)}$$

$$\boxed{y(x) = \frac{w_o}{24EI}(-x^4 + 4\ell x^3 - 6\ell^2 x^2)}$$

Solve this example with x from right to left.

Which one is easier ?! Why?

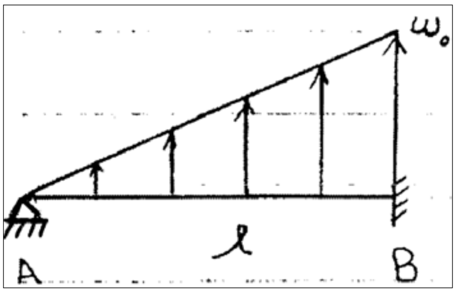


<u>Example 5</u>:

For the beam shown, determine the reactions.

$\boxed{EI = Constant}$

Note that forces/reactions in the x-direction are usually ignored in beams.



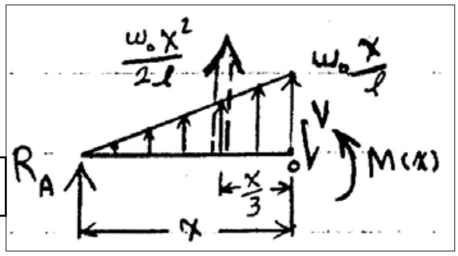
<u>Solution</u>:

Since the beam is ***<u>statically indeterminate</u>***, the **reactions are not known** and, thus, the ***moment equation can <u>not</u> be written explicitly***; therefore, it has to be **written in terms of some of the unknown reactions**. $\Rightarrow$

$+\circlearrowleft \sum M_o = 0 \Rightarrow M(x) - R_A x - \left(\frac{w_o x^2}{2\ell}\right)\left(\frac{x}{3}\right) = 0 \Rightarrow$

$$M(x) = R_A x + \frac{w_o}{6\ell}x^3$$

FBD



$$EI\theta(x) = \int M(x)dx = \int (R_A x + \frac{w_o}{6\ell}x^3)dx = \frac{1}{2}R_A x^2 + \frac{1}{24\ell}w_o x^4 + C_1$$

$$EIy(x) = \int EI\theta(x)dx = \int \left(\frac{1}{2}R_A x^2 + \frac{1}{24\ell}w_o x^4 + C_1\right)dx$$

$$= \frac{1}{6}R_A x^3 + \frac{1}{120\ell}w_o x^5 + C_1 x + C_2$$

B.C.'s: $y(0) = 0$; $\theta(\ell) = 0$; $y(\ell) = 0 \Rightarrow$

3 B.C.'s & 3 unknowns (C_1, C_2, R_A) $\Rightarrow$ <u>ok</u>

y(0) = 0 $\Rightarrow$ C_2 = 0

$$\theta(\ell)=0 \quad \Rightarrow \quad \frac{1}{2} R_A \ell^2 + \frac{w_o}{24\ell}\ell^4 + C_1 = 0 \quad \Rightarrow$$

$$\frac{1}{2}\ell^2 R_A + C_1 + \frac{w_o}{24}\ell^3 = 0 \qquad \textbf{(1)}$$

$$y(\ell)=0 \quad \Rightarrow \quad \frac{1}{6} R_A \ell^3 + \frac{w_o}{120\ell}\ell^5 + C_1\ell = 0 \quad \Rightarrow$$

$$\frac{1}{6}\ell^2 R_A + C_1 + \frac{w_o}{120}\ell^3 = 0 \qquad (2)$$

By solving Equations (1) & (2),

$$R_A = -\frac{w_o \ell}{10} \quad \Rightarrow \qquad \boxed{R_A = \frac{w_o\ell}{10}} \quad (\downarrow)$$

and $\quad C_1 = \dfrac{w_o \ell^3}{120}$

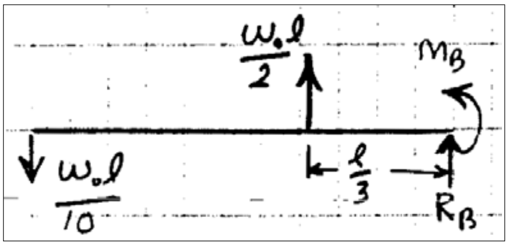
$$\Rightarrow \quad \theta(x) = \frac{w_o}{120\ell EI}\,(5x^4 - 6\ell^2 x^2 + \ell^4)$$

$$y(x) = \frac{w_o}{120\ell EI}\,(x^5 - 2\ell^2 x^3 + \ell^4 x)$$

At this stage, static can be used to find the remaining reactions. In the FBD,

$+\uparrow \sum F_y = 0 \quad \Rightarrow$

FBD



$$\frac{w_o\ell}{2} - \frac{w_o\ell}{10} + R_B = 0$$

$$\Rightarrow \quad R_B = -\frac{2}{5} w_o \ell \quad \Rightarrow \qquad \boxed{R_B = \frac{2 w_o \ell}{5}} \quad (\downarrow)$$

$$\circlearrowleft + \sum M_B = 0 \quad \Rightarrow \quad \frac{w_o\ell^2}{10} - \frac{w_o\ell^2}{6} + M_B = 0 \quad \Rightarrow \quad \boxed{M_B = \frac{w_o\ell^2}{15}} \quad (\circlearrowleft)$$

[We can also use the relation M_B = - M(ℓ).]   (Why?!)

2) Successive integration method:

This method is similar to the double integration procedure except that it **starts with the load equation** instead of the moment equation. ***This method is utilized when the loading on the beam is so complicated that it is not easy to obtain the moment equation***. Otherwise, double integration method is better. Note that **4 constants**, not 2, appear after integrating the load function four times. Thus, 4 B.C.'s are needed; they include shear & moment B.C.'s

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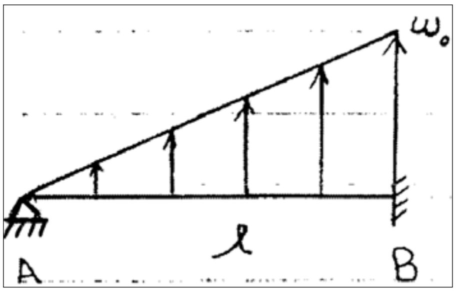
Example 6:

Rework Example 5 utilizing the successive integration method.

Solution:

EI = Constant



Note that forces/reactions in the x-direction are usually ignored in beams.

$$w(x) = \frac{w_o}{\ell}\, x$$

$$V(x) = \int w\, dx = \frac{w_o}{2\ell} x^2 + C_1$$

$$M(x) = \int V\, dx = \frac{w_o}{6\ell} x^3 + C_1 x + C_2$$

[*Note that no need for FBD to obtain M(x)*]

$$EI\,\theta(x) = \int M\, dx = \frac{w_o}{24\ell} x^4 + \frac{C_1}{2} x^2 + C_2\, x + C_3$$

$$EI\, y(x) = \int EI\theta\, dx = \frac{w_o}{120\ell} x^5 + \frac{C_1}{6} x^3 + \frac{C_1}{2} x^2 + C_3 x + C_4$$

B.C.'s:

$M(0) = 0$; $y(0) = 0$; $\theta(\ell) = 0$; $y(\ell) = 0$ (4 equations & 4 unknowns $\Rightarrow$ ok)

$M(0) = 0 \Rightarrow C_2 = 0$

$y(0) = 0 \Rightarrow C_4 = 0$

$\theta(\ell) = 0 \Rightarrow$ $$\frac{w_o}{24}\ell^3 + \frac{C_1}{2}\ell^2 + C_3 = 0$$

$y(\ell) = 0 \Rightarrow$ $$\frac{w_o}{120}\ell^4 + \frac{C_1}{6}\ell^3 + C_3\,\ell = 0$$

two equations & 2 unknowns (C_1 and C_3) $\Rightarrow$

$$C_1 = -\frac{w_o \ell}{10}$$ and

$$C_3 = \frac{w_o \ell^3}{120} \quad \Rightarrow$$

$$\theta(x) = \frac{w_o}{120 \ell EI}(5x^4 - 6\ell^2 x^2 + \ell^4)$$

$$y(x) = \frac{w_o}{120 \ell EI}(x^5 - 2\ell^2 x^3 + \ell^4 x)$$

$R_A = V(0)$ $\Rightarrow$

$$\boxed{R_A = -\frac{w_o \ell}{10} = \frac{w_o \ell}{10} \ (\downarrow)} \quad \Rightarrow$$

***From equilibrium* (as in Example 5),**

$+\uparrow \sum F_y = 0$ & $+\circlearrowleft \sum M_B = 0$ $\Rightarrow$

$$\boxed{R_B = \frac{2w_o \ell}{5}} \ (\downarrow)$$ and

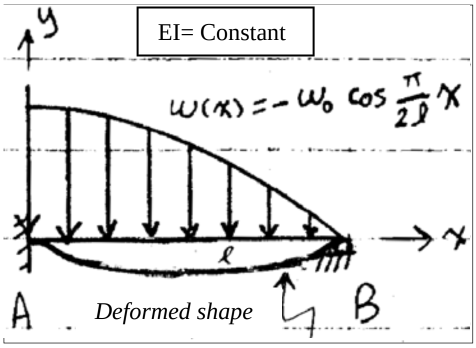
$$\boxed{M_B = \frac{w_o \ell}{15}} \ (\circlearrowleft)$$

[You can also use $R_B = -V(\ell)$ & $M_B = -M(\ell)$.]

(Why?!)

Example 7:

Obtain formulas for the slope and deflection, and determine the reactions at A and B for the beam shown.



Solution:

It is SI. (Why?! Show!)

$$w(x) = -w_o \cos\frac{\pi}{2\ell}x$$

$$V(x) = \int w\,dx = -w_o\left(\frac{2\ell}{\pi}\right)\sin\frac{\pi}{2\ell}x + C_1$$

$$M(x) = \int V\,dx = w_o\left(\frac{2\ell}{\pi}\right)^2\cos\frac{\pi}{2\ell}x + C_1\,x + C_2$$

$$EI\,\theta(x) = \int M\,dx + w_o\left(\frac{2\ell}{\pi}\right)^3\sin\frac{\pi}{2\ell}x + \frac{C_1}{2}x^2 + C_2\,x + C_3$$

$$EI\,y(x) = \int EI\,\theta\,dx = -w_o\left(\frac{2\ell}{\pi}\right)^4\cos\frac{\pi}{2\ell}x + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4$$

B.C.'s

$$\theta(0) = 0 \quad\Rightarrow\quad C_3 = 0$$

$$y(0) = 0 \quad\Rightarrow\quad -w_o\left(\frac{2\ell}{\pi}\right)^4 + C_4 = 0 \quad\Rightarrow\quad C_4 = w_o\left(\frac{2\ell}{\pi}\right)^4$$

$M(\ell) = 0 \quad\Rightarrow\quad C_1\ell + C_2 = 0$ **(1)**

$$y(\ell) = 0 \quad\Rightarrow\quad C_1\frac{\ell^3}{6} + C_2\frac{\ell^2}{2} + w_o\left(\frac{2\ell}{\pi}\right)^4 = 0 \qquad (2)$$

Two equations and two unknowns $\Rightarrow$

$$C_1 = \frac{48\ell}{\pi^4}w_o \qquad ; \qquad C_2 = -\frac{48\ell^2}{\pi^4}w_0 \quad\Rightarrow$$

$$\boxed{V(x) = -\left(\frac{2\ell}{\pi}\right)w_o\sin\frac{\pi}{2\ell}x + \frac{48\ell}{\pi^4}w_o}$$

$$\boxed{M(x)=\left(\frac{2\ell}{\pi}\right)^2 w_o\ \cos\ \frac{\pi}{2\ell}x+\frac{48\ell}{\pi^4}w_o\ x-\frac{48\ell^2}{\pi^4}w_o}$$

$$\boxed{EI\,\theta(x)=\left(\frac{2\ell}{\pi}\right)^3 w_o\ \sin\ \frac{\pi}{2\ell}x+\frac{24\ell}{\pi^4}w_o\ x^2-\frac{48\ell^2}{\pi^4}w_o\ x}$$

$$\boxed{EI\ y(x)=-\left(\frac{2\ell}{\pi}\right)^4 w_o\ \cos\ \frac{\pi}{2\ell}x+\frac{8\ell}{\pi^4}w_o\ x^3-\frac{24\ell^2}{\pi^4}w_o\ x^2+\left(\frac{2\ell}{\pi}\right)^4 w_o}$$

$R_A=V(0)\ \Rightarrow$ $\boxed{R_A=\frac{48\ell}{\pi^4}w_o}$ $(\uparrow)$

$M_A=M(0)\qquad\Rightarrow$

$$\boxed{M_A=\left(\frac{4\pi^2-48}{\pi^4}\right)\ell^2 w_o=-0.0875\,\ell^2\,w_o=0.0875\,\ell^2 w_o}\quad (\circlearrowleft)$$

[*Note direction of M_A ! Why??!!*]

$R_B=-V(\ell)\qquad\Rightarrow$

$$\boxed{R_B=\left(\frac{2}{\pi}-\frac{48}{\pi^4}\right)\ell\,w_o=0.144\,\ell\,w_o}\quad (\uparrow)$$

Can you use double integration method to solve this example?! Explain!

3) Singularity function method:

The <u>singularity functions permit the expression of ANY system of loads as an equivalent distributed load</u>. Thus, <u>one equation</u> for each of w, V , M, θ, and y can be written.

Macaulay functions:

$$\langle x-a\rangle^n = \begin{cases} 0 & x<a \\ (x-a)^n & x \geq a \end{cases} \qquad \qquad n = 0,1,2,3 \ldots$$

$$\int_o^x \langle \xi - a\rangle^n \, d\xi = \frac{\langle x-a\rangle^{n+1}}{n+1}$$

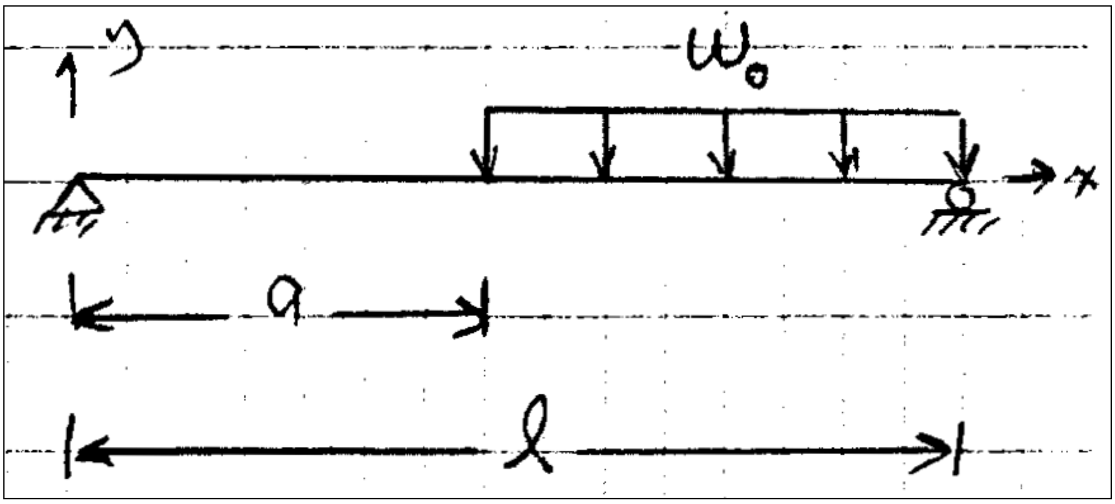
Note that ⟨ ⟩ are called <u>pointed</u> (or angle) brackets.

Also note that the quantity <u>inside</u> the brackets ⟨…⟩ can <u>never be negative</u>. If you “tried it” and it came out to be negative, then it means it is <u>ZERO</u>.

<u>Example 8:</u>

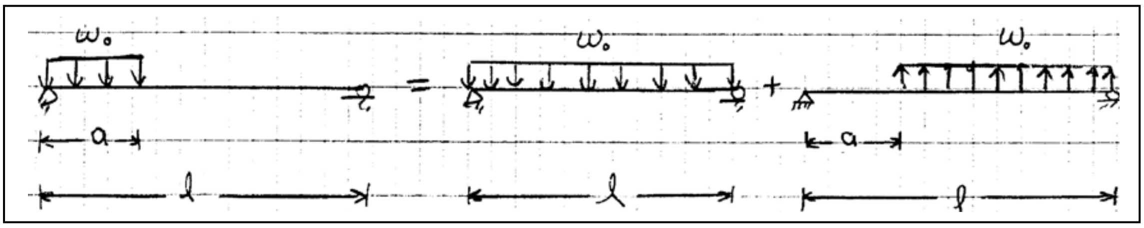
Using the singularity function, write the equivalent load equation for each of the beams shown below.

(a)



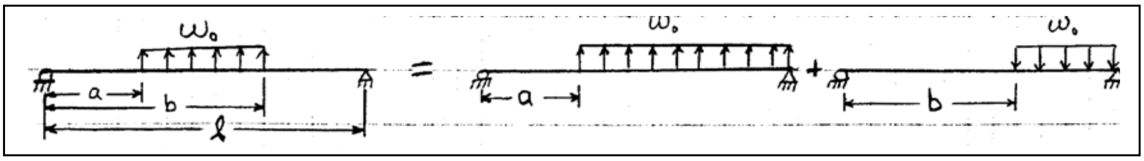
$$w_e = -\, w_o \, \langle x-a\rangle^0$$

(b)



$$w_e = - w_o \langle x\text{-}0\rangle^0 + w_o \langle x\text{-}a\rangle^0$$

(c)



$$w_e = w_o \langle x\text{-}a\rangle^0 - w_o \langle x\text{-}b\rangle^0$$

Singularity functions for <u>concentrated force</u>

$$\langle x-a\rangle^{-1} = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

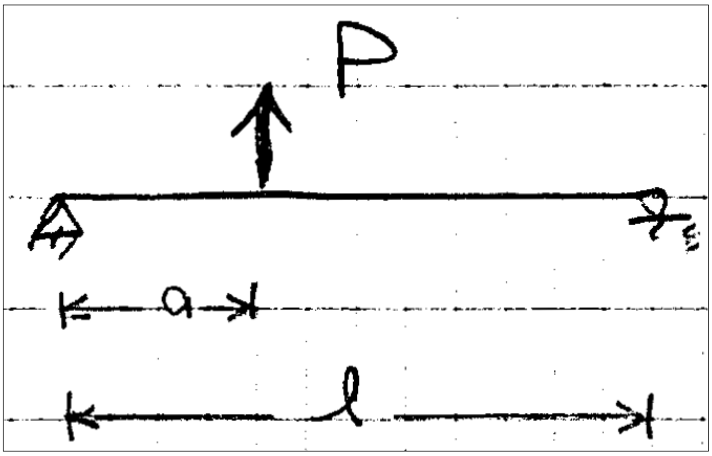
$$\int_o^x \langle \xi - a\rangle^{-1}\, d\xi = \langle x-a\rangle^0$$

(Dirac Delta or unit impulse function)

$$w_e = P \ \langle x\text{-}a\rangle^{-1}$$

$$\int_o^x w_e\, d\xi = \int_o^x P\langle \xi - a\rangle^{-1}\, d\xi$$

$$= P\langle x-a\rangle^0 \equiv P$$

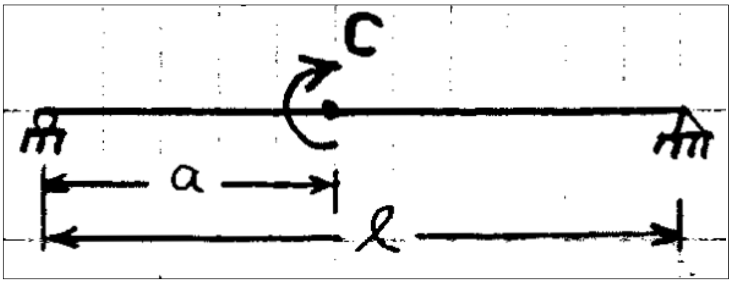


Singularity functions for <u>concentrated couple</u>

$$\langle x-a\rangle^{-2} = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

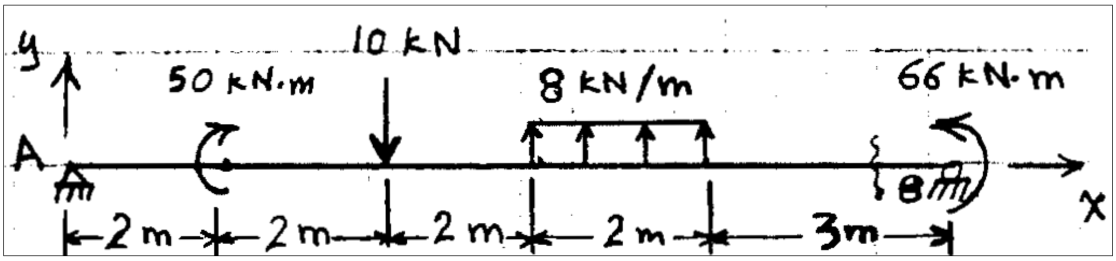
$$\int_{0}^{x} \langle \xi - a\rangle^{-2}\, d\xi = \langle x-a\rangle^{-1}$$

$w_e = C\,\langle x\text{-}a\rangle^{-2}$



<u>Example 9:</u>

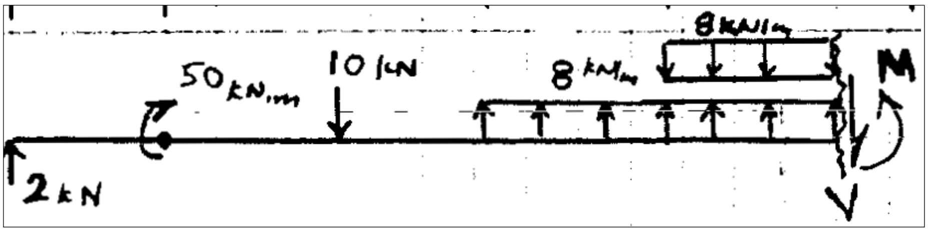
Write a <u>single</u> equation for M using the singularity function.



Solution:

R_A = 2 kN ↑ ; R_B = 8 kN ↓

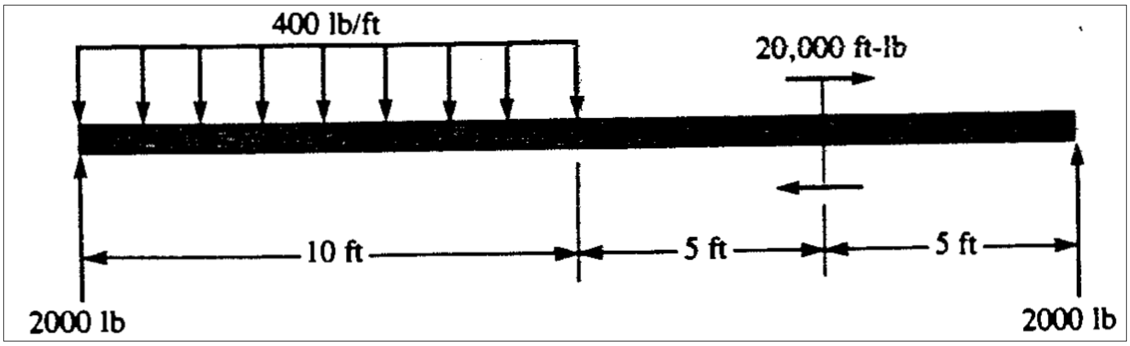
Draw FBD of the <u>last</u> segment ⇒



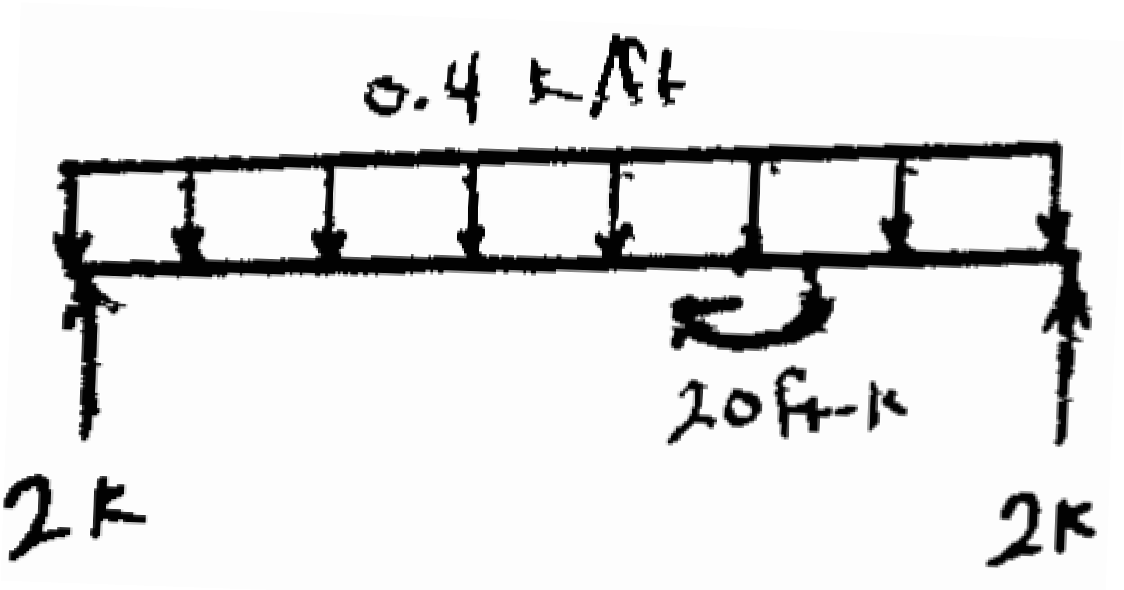
$$M(x) = 2\langle x-0\rangle^{1} + 50\langle x-2\rangle^{0} - 10\langle x-4\rangle^{1} + 4\langle x-6\rangle^{2} - 4\langle x-8\rangle^{2}$$

Example 10:

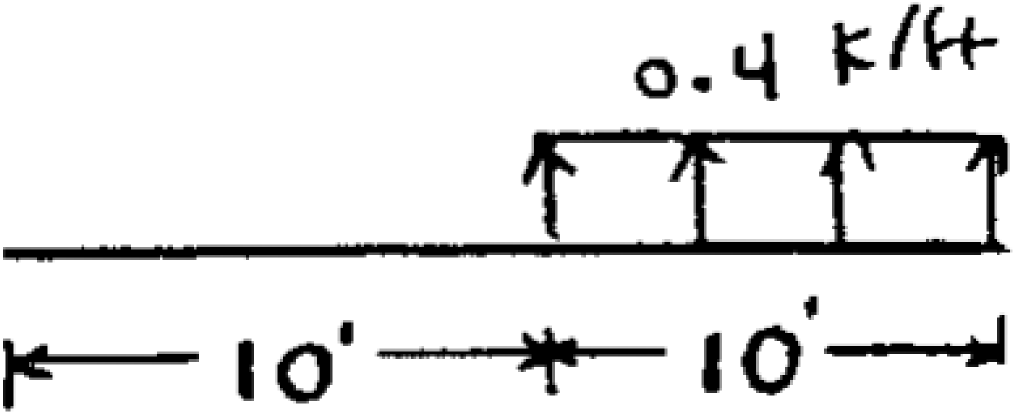
Determine the equivalent distributed load associated with the beam shown in the figure below. Determine the shear, moment, slope, and deflection equations, using the Macaulay functions and the singularity functions.



‖



+



$$w_e = 2\langle x-0\rangle^{-1} - \left[0.4\langle x-0\rangle^{0} - 0.4\langle x-10\rangle^{0}\right] + 20\langle x-15\rangle^{-2} + 2\langle x-20\rangle^{-1}$$
(a)

The shear force equation is obtained by integrating Eq. (a); consequently,

$$V(x) = 2\langle x-0\rangle^{0} - \left[0.4\langle x-0\rangle^{1} - 0.4\langle x-10\rangle^{1}\right] + 20\langle x-15\rangle^{-1} + 2\langle x-20\rangle^{0}$$
(b)

The moment equation is obtained by integrating Eq. (b); thus

$$M(x) = 2\langle x-0\rangle^{1} - \left[0.2\langle x-0\rangle^{2} - 0.2\langle x-10\rangle^{2}\right] + 20\langle x-15\rangle^{0} + 2\langle x-20\rangle^{1}$$
(c)

Notice that neither equation requires a constant of integration because we included the reactions in the expression for the equivalent distributed load. If the reactions had not been included in w_e , a constant of integration would be required for each integration.

The equations for slope and deflection follow from Eq. (c):

$$EI\,\theta(x) = \langle x-0\rangle^{2} - \left[\frac{0.2}{3}\langle x-0\rangle^{3} - \frac{0.2}{3}\langle x-10\rangle^{3}\right] + 20\langle x-15\rangle^{1} + \langle x-20\rangle^{2} + C_1 \qquad \text{(d)}$$

and

$$EI\,y(x) = \tfrac{1}{3}\langle x-0\rangle^{3} - \left[\frac{0.2}{12}\langle x-0\rangle^{4} - \frac{0.2}{12}\langle x-10\rangle^{4}\right] + 10\langle x-15\rangle^{2} + \tfrac{1}{3}\langle x-20\rangle^{3} + C_1 x + C_2 \qquad \text{(e)}$$

A constant of integration has been included for each integration that leads to the last two equations. These constants are required so that boundary conditions appropriate to the problem can be satisfied. In the present case, the boundary conditions yield

$y(0) = 0 \quad \Rightarrow \quad C_2 = 0$ **(f)**

$$y(20) = 0 \qquad \Rightarrow$$

$$\frac{20^3}{3} - \left[\frac{0.2}{12}(20)^4 - \frac{0.2}{12}(10)^4\right] + 10(5)^2 + 20C_1 = 0 \qquad \textbf{(g)}$$

Accordingly,

$$C_1 = -\frac{500}{24} \tag{h}$$

Let us write the shear and moment equations for the intervals $0 \le x \le 10$ and $10 \le x \le 15$. From Eqs. (b) and (c), we determine that

$$\left.\begin{array}{ccc} 0 \le x \le 10 & & 10 \le x \le 15 \\ V(x) = 2 - 0.4x & \text{and} & V(x) = 2 - 0.4x + 0.4(x-10) \\ M(x) = 2x - 0.2x^2 & & M(x) = 2x - 0.2x^2 + 0.2(x-10)^2 \end{array}\right\} \tag{i}$$

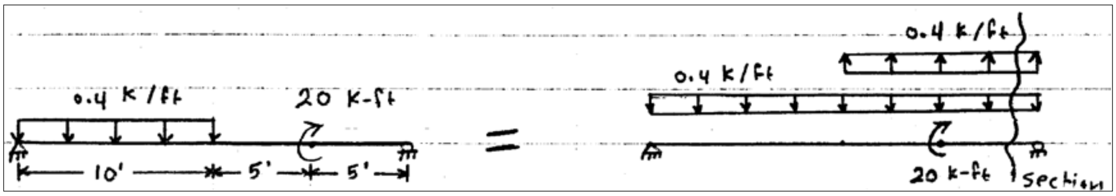
Verify that these equations are correct by drawing appropriate free-body diagrams and invoking force and moment equilibrium.

Example 11:

Rework Example 10 above by starting with the moment equation.

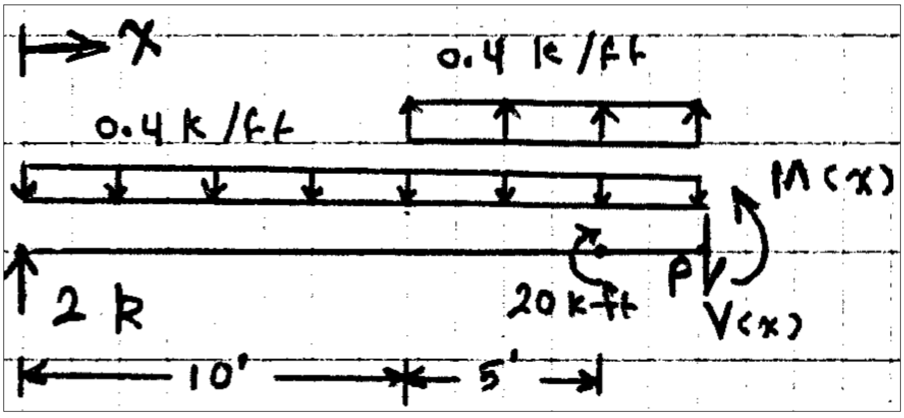
Solution:

Note that once the distributed load starts, it has to continue up to the end of the beam. Thus, the load is redrawn as shown.



Note the directions of forces

Next, make a section (cut) through the <u>last</u> segment of the beam ("near" the right support) after calculating the reactions. Then, draw the FBD of the left portion as shown.



Now, the moment equation can be written.

$$\sum M_p = 0 \quad \Rightarrow$$

$$M(x)=2\langle x-0\rangle^1 - \frac{0.4}{2}\langle x-0\rangle^2 + \frac{0.4}{2}\langle x-10\rangle^2 + 20\langle x-15\rangle^0$$

(Note that the right reaction is <u>not</u> involved in the equation. Why?!)

$$EI\,\theta(x)=\langle x-0\rangle^2 - \frac{0.4}{2(3)}\langle x-0\rangle^3 + \frac{0.4}{2(3)}\langle x-10\rangle^3 + 20\langle x-15\rangle^1 + C_1$$

$$EI\,y(x) = \frac{1}{3}\langle x-0\rangle^3 - \frac{0.4}{24}\langle x-0\rangle^4 + \frac{0.4}{24}\langle x-10\rangle^4 + \frac{20}{2}\langle x-15\rangle^2 + C_1\,X + C_2$$

B.C.'s: $y(0) = 0 \Rightarrow C_2 = 0$

$y(20) = 0 \Rightarrow C_1 = -125/6$

Singularity Function:

$$EI\,y(x)=\frac{1}{3}\langle x-0\rangle^3 - \frac{1}{60}\langle x-0\rangle^4 + \frac{1}{60}\langle x-10\rangle^4 + 10\langle x-15\rangle^2 - \frac{125}{6}\langle x-0\rangle$$

[for any value of x]

Normal Functions:

$$EI\,y(x)=\frac{1}{3}x^3 - \frac{1}{60}x^4 - \frac{125}{6}x \quad \Leftarrow \text{ for } 0 \le x \le 10'$$

$$EIy(x)=\frac{1}{3}x^3 - \frac{1}{60}x^4 + \frac{1}{60}(x-10)^4 - \frac{125}{6}x \quad \Leftarrow \text{ for } 10' \le x \le 15'$$

$$EI\,y(x) = \frac{1}{3}x^3 - \frac{1}{60}x^4 + \frac{1}{60}(x-10)^4 + 10(x-15)^2 - \frac{125}{6}x$$

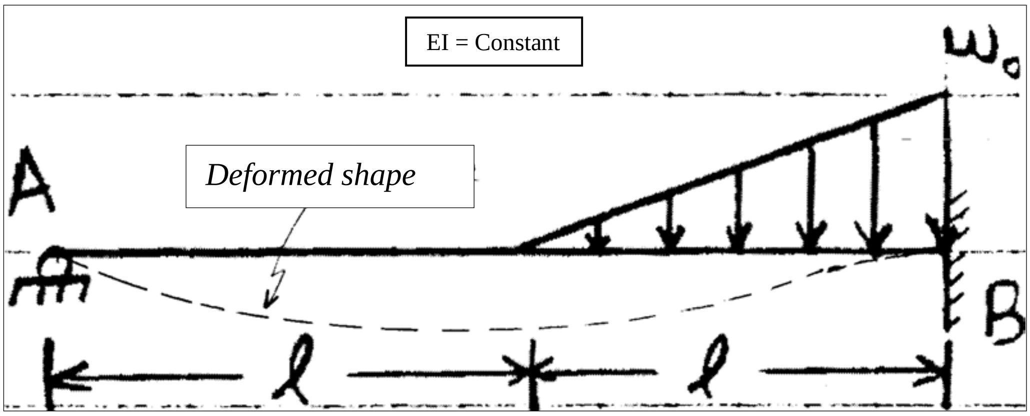
for $15' \le x \le 20'$

Some of the equations above can be simplified.

Example 12

Given:

The beam shown

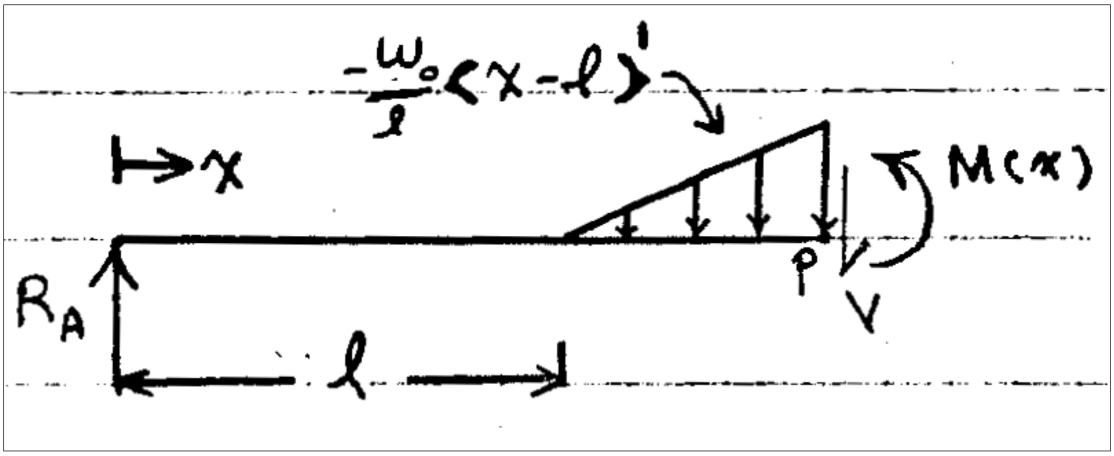


Required.:

The reaction at A

Solution:

Since the beam is *statically indeterminate*, the moment equation must be expressed in terms of some of the unknown reactions.



From the FBD

$$\sum M_p = 0 \quad \Rightarrow \quad M(x) = R_A \langle x-0\rangle^1 - \frac{w_o}{\ell}\langle x-\ell\rangle^1 \left(\frac{\langle x-\ell\rangle^1}{2}\right)\left(\frac{\langle x-\ell\rangle^1}{3}\right)$$

$$= R_A \langle x-0\rangle^1 - \frac{w_o}{6\ell}\langle x-\ell\rangle^3$$

$$EI\,\theta(x) = \int M(x)dx = \frac{1}{2}R_A\left\langle x-0\right\rangle^2 - \frac{w_o}{24\ell}\left\langle x-\ell\right\rangle^4 + C_1$$

$$EI\,y(x) = \int EI\,\theta(x)dx = \frac{1}{6}R_A\left\langle x-0\right\rangle^3 - \frac{w_o}{120\ell}\left\langle x-\ell\right\rangle^5 + C_1\,x + C_2$$

B.C's.:

$y(0) = 0$

$\theta(2\ell) = 0$

$y(2\ell) = 0$

$\Rightarrow$ ***3 equations & 3 unknowns (C_1, C_2, and R_A)***

$y(0) = 0 \quad \Rightarrow \quad C_2 = 0$

$$\theta(2\ell) = 0 \;\Rightarrow\; \frac{1}{2}R_A(2\ell)^2 - \frac{w_o}{24\ell}(2\ell-\ell)^4 + C_1 = 0 \;\Rightarrow$$

$$C_1 = \frac{w_o}{24}\ell^3 - 2R_A\ell^2$$

$$y(2\ell) = 0 \;\Rightarrow\; \frac{1}{6}R_A(2\ell)^3 - \frac{w_o}{120\ell}(2\ell-\ell)^5 + C_1(2\ell) = 0$$

$$\Rightarrow\; \frac{4}{3}R_A\ell^3 - \frac{w_o}{120}\ell^4 + \frac{w_o}{12}\ell^4 - 4R_A\ell^3 = 0 \quad \Rightarrow$$

$$\boxed{R_A = \frac{9}{320}w_o\ell}$$

Note that once R_A is found, the remaining reactions at B can be determined by the Statics equilibrium equations. (Try it yourself !)