#4

Examples : Statically - Indeterminate Axially - Loaded Members

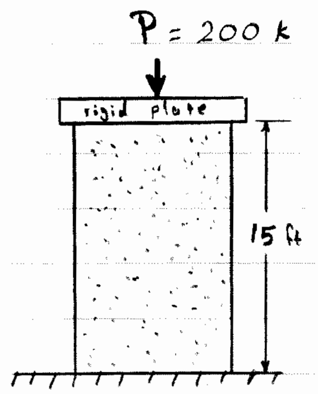
Example 1 :

Given :

The reinforced concrete column shown

$E_s = 30000$ ksi

$E_c = 3000$ ksi

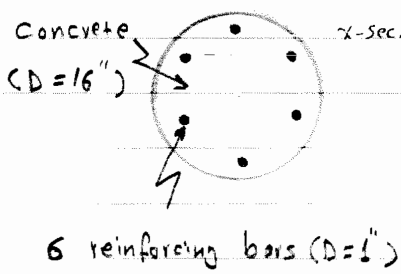


Req.d. :

The stresses in the steel & concrete

First, "estimate" the answers!

Soln. :



① Equilibrium :

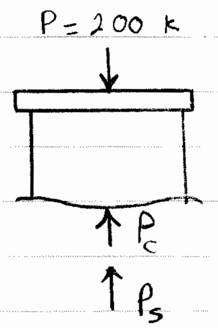
$$P_c + P_s - 200 = 0$$

② Geometric compatibility :

$$e_c = e_s$$

③ Material behavior :

$$e = Pl/EA \qquad \text{(derived before based on } \sigma = E\varepsilon\text{)}$$



From ③ into ②,

$$\left(\frac{Pl}{EA}\right)_c = \left(\frac{Pl}{EA}\right)_s$$

$$A_s = 6\left(\frac{\pi}{4}\right)(1)^2 = 1.5\pi \text{ in}^2$$

$$A_c = \frac{\pi}{4}(16)^2 - \underbrace{1.5\pi}_{\text{it can be neglected}} = 62.5\pi \text{ in}^2$$

$$\Rightarrow \left(\frac{-P(15\times12)}{3000(62.5\pi)}\right)_c = \left(\frac{-P(15\times12)}{30000(1.5\pi)}\right)_s \Rightarrow 15 P_c = 62.5 P_s$$

$$\Rightarrow P_c = \frac{62.5}{15} P_s \quad ④$$

from ④ into ① , $\frac{62.5}{15} P_s + P_s = 200 \Rightarrow P_s = 38.71$ k

$\Rightarrow P_c = 161.3$ k

$$\sigma = \frac{P}{A} \Rightarrow \sigma_s = \frac{38.71}{1.5\pi} \Rightarrow \boxed{\sigma_s = 8.21 \text{ ksi}}$$

$$\sigma_c = \frac{161.3}{62.5\pi} \Rightarrow \boxed{\sigma_c = 0.821 \text{ ksi}}$$

• Note that $\frac{P}{A}$ can be replaced by $\sigma \Rightarrow$ get σ directly.

Example 2 :

Given :

As in Example 1

Req'd. :

The maximum force P which can be applied if the allowable stresses are :

$$\sigma_s = 15 \text{ ksi}$$
$$\sigma_c = 1.2 \text{ ksi}$$

* Guess the answer!

Sol'n. :

① Equilibrium : $P_c + P_s = P_{max}$

② Compatibility : $e_c = e_s$

④ Material behavior : $e = Pl/EA$

$$\Rightarrow \left(\frac{P}{A}\frac{l}{E}\right)_c = \left(\frac{P}{A}\frac{l}{E}\right)_s \Rightarrow \left(\sigma\frac{l}{E}\right)_c = \left(\sigma\frac{l}{E}\right)_s \quad \Leftarrow l_c = l_s$$

$$\frac{\sigma_c}{3000} = \frac{\sigma_s}{30000} \Rightarrow \sigma_s = 10\,\sigma_c$$

Now, we assume σ_s or σ_c controls (i.e., one of them will reach the allowable stress before the other. It is unlikely that both will reach the allowable stresses at the same time; in such a case, this is called optimum design.)

Let's assume that σ_c controls (Why ?!)

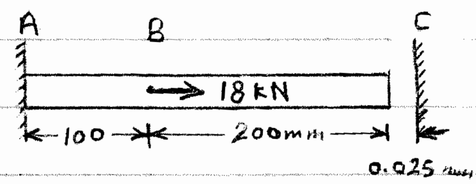
$$\Rightarrow \quad \sigma_c \equiv 1.2 \text{ ksi}$$
$$\Rightarrow \quad \sigma_s = 10(1.2) = 12 \text{ ksi} < 15 \text{ ksi} \Rightarrow \underline{\underline{\text{ok}}}$$
$$\Rightarrow \quad P_c = \sigma A = 1.2(62.5\pi) = 235.6 \text{ k}$$
$$P_s = \sigma A = 12(1.5\pi) = 56.55 \text{ k}$$
$$P_{max} = P_c + P_s = 235.6 + 56.55 \Rightarrow \boxed{P_{max} = 292 \text{ k}}$$

Is the answer "reasonable"? Why? THINK!

Example 3:

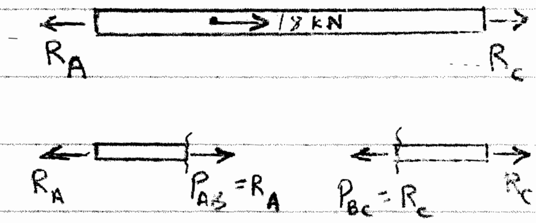
Given:

The figure shown



Aluminum

$A = 500\ mm^2$



Req'd.:

The reactions and stresses [[Guess first!]]

Sol'n.:

First, check if the gap closes:

$$e = \frac{PL}{EA} = \frac{18\,(100)}{70\,(500)} = 0.0514\ mm > 0.025 \Rightarrow \text{the gap closes} \Rightarrow \text{stat. indet.}$$

① Equilibrium:

$$18 + R_c - R_A = 0 \qquad \boxed{1}$$

② Geometric compatibility:

$$e_{AB} + e_{Bc} = 0.025$$

③ Material behavior

$$\left(\frac{PL}{EA}\right)_{AB} + \left(\frac{PL}{EA}\right)_{Bc} = 0.025 \Rightarrow \frac{R_A\,(100)}{70\,(500)} + \frac{R_c\,(200)}{70(500)} = 0.025$$

$$\Rightarrow R_A + 2R_c = (0.025)(350) = 8.75 \qquad \boxed{2}$$

from eq. $\boxed{1}$ and $\boxed{2}$, $3R_c = -9.25 \Rightarrow$

$R_c = 3.083\ kN \leftarrow$
$R_A = 14.92\ kN \leftarrow$

$$\sigma_{AB} = \frac{14.92}{500} \Rightarrow \boxed{\sigma_{AB} = 29.83\ MPa\ (T)}$$

$$\sigma_{Bc} = \frac{3.083}{500} \Rightarrow \boxed{\sigma_{Bc} = 6.17\ MPa\ (C)}$$

"[illegible]" ? why? THINK!

Example 4:

Problem statement:

Rework Example 3 if the aluminum rod is heated by 30°c, in addition to the applied load.

Solution:

Method ①: Assume the load is applied first

From the previous example,

$R_A = 14.92$ kN ← due to the load only

$R_C = 3.083$ kN ← " " " " "

Now, consider ΔT only (remember that there is no gap now)

Take AC as one element (constant A and E, no load)

$$\delta_{AC} = \alpha \Delta T \ell = 23(10)^{-6}(30)(300) = 0.207 \text{ mm}$$

$$\delta_{AC} = \frac{R_C^T L}{EA}$$

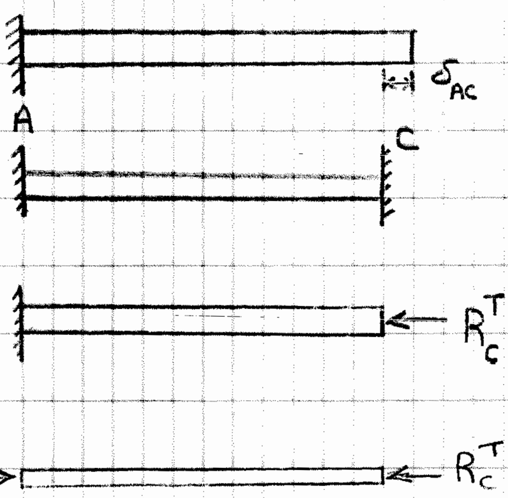
free expansion

$$0.207 = \frac{R_C^T (300)}{70(500)}$$

no δ allowed

$$\Rightarrow R_C^T = 24.15 \text{ kN} \leftarrow$$

reaction to prevent δ



from equilibrium,

$R_A^T = 24.15$ kN →

FBD

By the superposition principle,

$$R_A = R_A^T + R_A^{Load}$$

$$\Rightarrow R_A = 24.15 - 14.92 \Rightarrow \boxed{R_A = 9.23 \text{ kN} \rightarrow}$$

Similarly, $R_C = -24.15 - 3.083 \Rightarrow$ $\boxed{R_C = 27.23 \text{ kN} \leftarrow}$

$\sigma_{AB} = \frac{P_{AB}}{A} = \frac{9.23}{500} \Rightarrow$ $\boxed{\sigma_{AB} = 18.5 \text{ MPa (C)}}$

$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{27.23}{500} \Rightarrow$ $\boxed{\sigma_{BC} = 54.5 \text{ MPa (C)}}$

Method ②: Assume temperature is increased first

$\delta_{AC} = \alpha \Delta T L = 0.207$ mm (as before)

Now, the rod can expand freely by 0.025 mm

$\Rightarrow$ elongation to be prevented $= 0.207 - 0.025 = 0.182$ mm

$\Rightarrow 0.182 = \frac{R_C^T L}{EA}$ (as before except δ is different)

$\Rightarrow R_C^T = 21.23$ kN $\leftarrow$

$R_A^T = 21.23$ kN $\rightarrow$ (from equilibrium)

Now, Let's apply the load:

① Equilibrium:

$18 + R_C^L - R_A^L = 0$ $\boxed{1}$ (as before)

② Geometric compatibility:

$e_{AB} + e_{BC} = 0$ (as before except there is no gap since it was closed by the temperature first)

③ Material behavior:

$\left(\frac{PL}{EA}\right)_{AB} + \left(\frac{PL}{EA}\right)_{BC} = 0 \Rightarrow R_A^L + 2R_C^L = 0$ $\boxed{2}$

Solving eqs. $\boxed{1}$ and $\boxed{2}$ gives $R_A^L = 12$ kN $\leftarrow$

and $R_C^L = 6$ kN $\leftarrow$

By the principle of superposition,

$R_A = 21.23 - 12.0 \Rightarrow$ $\boxed{R_A = 9.23 \text{ kN} \rightarrow}$

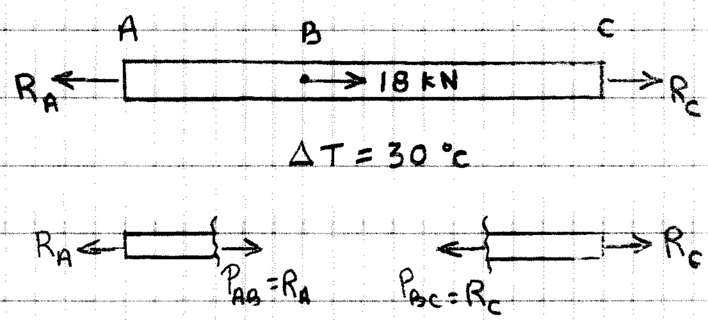
$R_c = -21.23 - 6.0 \Rightarrow$ $\boxed{R_c = 27.23 \text{ kN} \leftarrow}$

as before, $\boxed{\sigma_{AB} = 18.5 \text{ MPa (C)}}$ $\boxed{\sigma_{BC} = 54.5 \text{ MPa (C)}}$

Method ③ : Assume the load and temperature are applied simultaneously

① Equilibrium :

$-R_A + R_c = -18$ $\boxed{3}$



② Geometric Compatibility

$e_{AB} + e_{BC} = 0.025$

③ Material behavior

$$e_{AB} = \left(\frac{PL}{EA}\right)_{AB} + \alpha \Delta T L_{AB}$$
$$= \left(\frac{R_A(100)}{70(500)}\right) + 23(10)^{-6}(30)(100) = \frac{R_A}{350} + 0.069$$

$$e_{BC} = \left(\frac{PL}{EA}\right)_{BC} + \alpha \Delta T L_{BC} = \frac{R_c(200)}{70(500)} + 23(10)^{-6}(30)(200) = \frac{2R_c}{350} + 0.138$$

Thus,

$$\left(\frac{R_A}{350} + 0.069\right) + \left(\frac{2R_c}{350} + 0.138\right) = 0.025 \Rightarrow R_A + 2R_c = -63.7 \quad \boxed{4}$$

Solving eqs. $\boxed{3}$ and $\boxed{4}$ yields $\boxed{R_A = 9.23 \text{ kN} \rightarrow}$ and $\boxed{R_c = 27.23 \text{ kN} \leftarrow}$

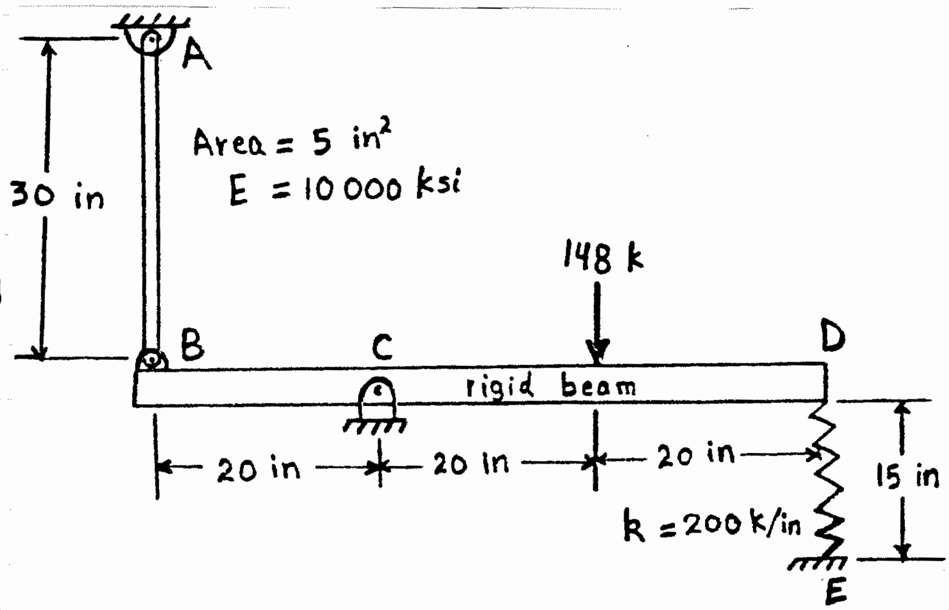
As before, $\sigma = \frac{P}{A} \Rightarrow$ $\boxed{\sigma_{AB} = 18.5 \text{ MPa (C)}}$ and $\boxed{\sigma_{BC} = 54.5 \text{ MPa (C)}}$

Of course, the answers from the three different methods are identical.
Which method is easier? It is up to you to decide !!

Example 5 :

Given :

The figure shown



Req.d. :

The force in the spring

The stress in rod AB

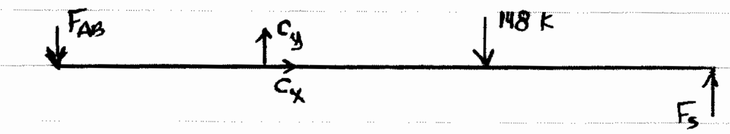
Soln. :

① Equilibrium :

$\Sigma M_c = 0 \Rightarrow$

$20\,F_{AB} + 40\,F_{sp} - 148(20) = 0$

$\Rightarrow F_{AB} + 2\,F_{sp} - 148 = 0$



② Compatibility :

$$\frac{e_{AB}}{20} = \frac{e_{sp}}{40}$$

$\Rightarrow e_{sp} = 2\,e_{AB}$

③ Material behavior :

$$\frac{-F_{sp}}{k} = 2\left(\frac{-P\ell}{EA}\right)_{AB}$$

⇐ Note that − signs are used because compression is assumed for both the spring and rod AB (by physical inspection). You may assume one or both as tension. (Take care of signs.)

$$\Rightarrow \frac{F_{sp}}{200} = 2\,\frac{30\,F_{AB}}{10000\,(5)}$$

$\Rightarrow F_{sp} = 0.24\,F_{AB}$

From ③ into ①,

$F_{AB} + 2\,(0.24\,F_{AB}) = 148 \Rightarrow F_{AB} = 100\text{ k}$

$\sigma = P/A \Rightarrow \sigma_{AB} = 100/5 \Rightarrow$ $\boxed{\sigma_{AB} = 20\text{ ksi} \quad (C)}$

$F_{sp} = 0.24\,F_{AB} = 0.24\,(100) \Rightarrow$ $\boxed{F_{sp} = 24\text{ k} \quad (C)}$

Answers OK ? THINK !!

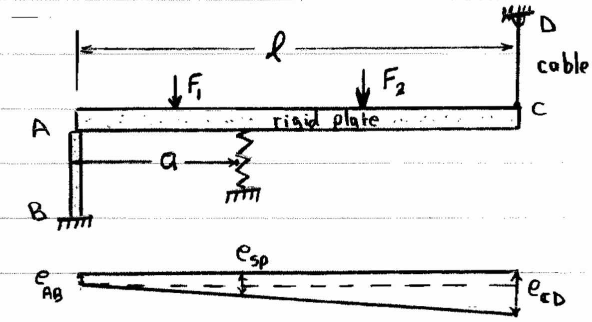
Example 6 :

Given :

The structures shown below

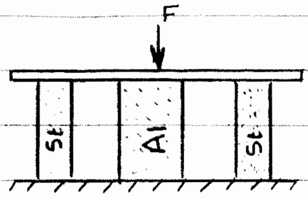
Req'd. :

The compatibility equations

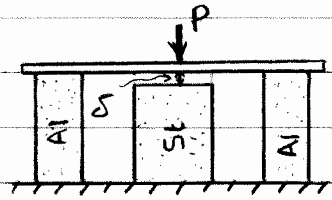


$$\frac{e_{sp}-e_{AB}}{a} = \frac{e_{CD}-e_{AB}}{\ell}$$ ⟸ note that all move

Soln. :

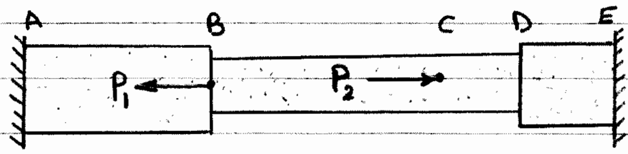


$$e_{st} = e_{Al}$$

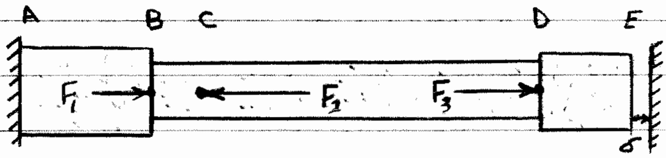


$$e_{Al} = e_{st} + \delta$$

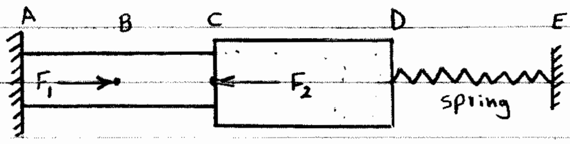
note δ before applying P



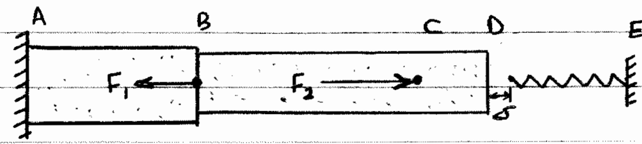
$$e_{AB} + e_{BC} + e_{CD} + e_{DE} = 0$$



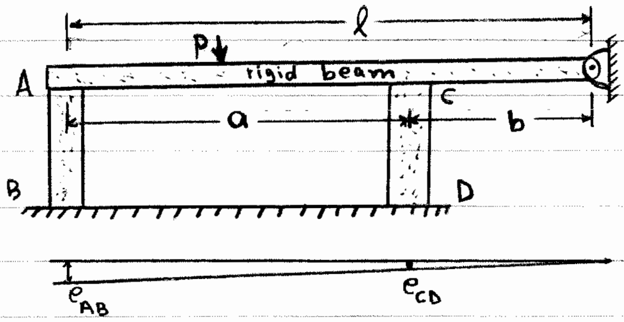
$$e_{AB} + e_{BC} + e_{CD} + e_{DE} = \delta$$



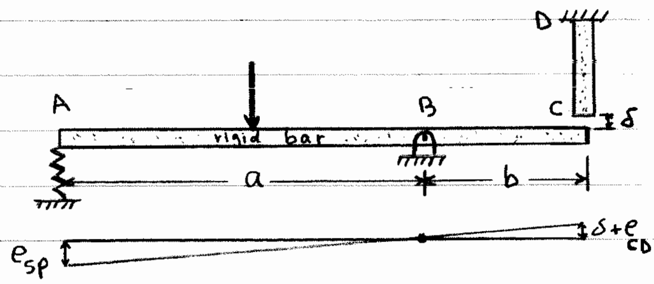
$$e_{AB} + e_{BC} + e_{CD} + e_{spring} = 0$$



$$e_{AB} + e_{BC} + e_{CD} + e_{spring} = \delta$$



$$\frac{e_{AB}}{\ell} = \frac{e_{CD}}{b}$$



$$\frac{e_{sp}}{a} = \frac{\delta + e_{CD}}{b}$$

very important notes : (1) The forces P and F_{sp} in $\frac{PL}{EA}$ and $\frac{F_{sp}}{k}$ are internal (get them from FBD's).

(2) The internal force can be assumed T or C.

(3) If the force is assumed Comp., then (−) sign must be used in the formulas PL/EA and F_{sp}/k.

(4) If there is a gap (δ), then check first if it closes or not: The problem becomes stat. indet. if it closes, and one of the compati. eq. above (or others) can be used. If not, the problem is stat. det. and no need for compati.