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The stress-strain relation for an infinitesimal material element can be written in matrix form as

$$\{\sigma\} = [D] \{\epsilon\} \quad (3.131)$$

where  $\{\sigma\}$  is the stress vector and  $\{\epsilon\}$  the strain vector, given by

$$\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \quad \{\epsilon\} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad (3.132)$$

and  $[D]$  is the *elastic material-stiffness matrix*, given in terms of  $\nu$  and  $E$  by

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \leftarrow 3-D \quad (3.133)$$

or alternatively, in terms of  $K$  and  $G$ , by

$$[D] = \begin{bmatrix} K + \frac{4}{3}G & K - \frac{2}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K + \frac{4}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K - \frac{2}{3}G & K + \frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (3.134)$$

The stress-strain relation (3.131) for the special case of the plane-stress condition ( $\sigma_z = \tau_{yz} = \tau_{zx} = 0$ ) takes the simple form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (3.135)$$

and the stress-strain relation (3.131) for the plane-strain condition ( $\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ ) takes the simple form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (1.136)$$

As for an axisymmetric case with  $\tau_{yz} = \tau_{zx} = \gamma_{yz} = \gamma_{zx} = 0$ , in the usual notation the stress-strain matrix relation (3.131) has the form

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} \quad (3.137)$$

$$K = \frac{E}{3(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

GENERALIZED STRESS-STRAIN MATRICES FOR ISOTROPIC MATERIALS AND THE PROBLEMS IN TABLE 3.1

Problem	Material Matrix C
Bar	$E$
Beam	$EI$
Plane Stress	$\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$
Plane strain	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$
Axisymmetric	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$
Three-dimensional	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & & & \\ & & & \frac{1-2\nu}{2(1-\nu)} & & \\ & & & & \frac{1-2\nu}{2(1-\nu)} & \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$ elements not shown are zeros
Plate bending	$\frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

Notation:  $E$  = Young's modulus,  $\nu$  = Poisson's ratio,  $t$  = thickness of plate,  $I$  = moment of inertia

CORRESPONDING KINEMATIC AND STATIC VARIABLES IN VARIOUS PROBLEMS

Problem	Displacement Components	Strain Vector $\epsilon^T$	Stress Vector $\tau^T$
Bar	$u$	$[\epsilon_{xx}]$	$[\tau_{xx}]$
Beam	$w$	$[\kappa_{xx}]$	$[M_{xx}]$
Plane stress	$u, v$	$[\epsilon_{xx} \ \epsilon_{yy} \ \gamma_{xy}]$	$[\tau_{xx} \ \tau_{yy} \ \tau_{xy}]$
Plane strain	$u, v$	$[\epsilon_{xx} \ \epsilon_{yy} \ \gamma_{xy}]$	$[\tau_{xx} \ \tau_{yy} \ \tau_{xy}]$
Axisymmetric	$u, v$	$[\epsilon_{xx} \ \epsilon_{yy} \ \gamma_{xy} \ \epsilon_{zz}]$	$[\tau_{xx} \ \tau_{yy} \ \tau_{xy} \ \tau_{zz}]$
Three-dimensional	$u, v, w$	$[\epsilon_{xx} \ \epsilon_{yy} \ \epsilon_{zz} \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx}]$	$[\tau_{xx} \ \tau_{yy} \ \tau_{zz} \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]$
Plate Bending	$w$	$[\kappa_{xx} \ \kappa_{yy} \ \kappa_{xy}]$	$[M_{xx} \ M_{yy} \ M_{xy}]$

Notation:  $\epsilon_x = \frac{\partial u}{\partial x}$ ,  $\epsilon_y = \frac{\partial v}{\partial y}$ ,  $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ , ...,  $\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}$ ,  $\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$ ,  $\kappa_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y}$ .