$\sigma - \varepsilon$ relationships

In $\boxed{\text{1-D}}$ linear elastic analysis, the relation between stress and strain is

$$\sigma = E\,\varepsilon \qquad \Leftarrow \text{Hooke's Law}$$

In 2-D and 3-D, what are the relations ?!!

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \overset{?}{\underset{!}{\rightleftarrows}} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

It takes sometime to derive these relations for different materials (as discussed in class).

However, for linear elastic isotropic materials, see the handout for:

3-D relations $\Leftarrow$ Generalized Hooke's Law

2-D relations $\Leftarrow$ Plane stress
$\qquad\qquad\quad\;\, \Leftarrow$ Plane strain

Other special relations $\Leftarrow$ Axisymmetric, plate, beam, ..etc

:call $\left\{ \sigma = E\,\varepsilon \right.$ ↳ elastic modulus or modulus of elasticity $\}$

$$\tau = G\,\gamma$$

↳ shear modulus or modulus of rigidity

$$G = \frac{E}{2(1+\nu)}$$ ← easy to prove but not required (ν = Poisson's ratio)

Volumetric Strain (e):

$$e = \frac{V_f - V_o}{V_o}$$ (V = Volume)

$$e \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$$ ← easy to prove

Bulk Modulus (K):

$$K = \frac{E}{3(1-2\nu)}$$

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$ ← σ_{mean} (ave σ)

$$\Rightarrow \sigma_m = K e$$ ⟷ $\left(K = \frac{\sigma_m}{e} \right)$

For incompressible materials (no change in volume), $e = 0$

$$\Rightarrow K \to \infty$$

$$\Rightarrow \underline{\underline{\nu = 0.5}}$$ ← for incomp. materials

#6 1/2

The stress-strain relation for an infinitesimal material element can be written in matrix form as

$$\{\sigma\} = [D]\{\epsilon\} \tag{3.131}$$

where $\{\sigma\}$ is the stress vector and $\{\epsilon\}$ the strain vector, given by

$$\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \qquad \{\epsilon\} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \tag{3.132}$$

and $[D]$ is the *elastic material-stiffness matrix*, given in terms of ν and E by

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{3.133}$$

← 3-D

or alternatively, in terms of K and G, by

$$[D] = \begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{3.134}$$

The stress-strain relation (3.131) for the special case of the *plane-stress* condition ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$) takes the simple form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{3.135}$$

and the stress-strain relation (3.131) for the *plane-strain* condition ($\epsilon_z = \gamma_{yz} = \gamma_{zx} = 0$) takes the simple form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{1.136}$$

As for an *axisymmetric case* with $\tau_{z\theta} = \tau_{r\theta} = \gamma_{z\theta} = \gamma_{\theta r} = 0$, in the usual notation the stress-strain matrix relation (3.131) has the form

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{zr} \end{bmatrix} \tag{3.137}$$

$$K = \frac{E}{3(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)}$$

Generalized Stress-Strain Matrices for Isotropic Materials and the Problems in Table 3.1

Problem	Material Matrix C
Bar	E
Beam	EI
Plane Stress	$\dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$
Plane strain	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$
Axisymmetric	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$
Three-dimensional	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & 1 & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 1 & & & \\ & & & \dfrac{1-2\nu}{2(1-\nu)} & & \\ & & & & \dfrac{1-2\nu}{2(1-\nu)} & \\ & & & & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$ elements not shown are zeros
Plate bending	$\dfrac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Notation: E = Young's modulus, ν = Poisson's ratio, t = thickness of plate, I = moment of inertia

Corresponding Kinematic and Static Variables in Various Problems

Problem	Displacement Components	Strain Vector $\boldsymbol{\epsilon}^T$	Stress Vector $\boldsymbol{\tau}^T$
Bar	u	$[\epsilon_{xx}]$	$[\tau_{xx}]$
Beam	w	$[\kappa_{xx}]$	$[M_{xx}]$
Plane stress	u, v	$[\epsilon_{xx}\ \ \epsilon_{yy}\ \ \gamma_{xy}]$	$[\tau_{xx}\ \ \tau_{yy}\ \ \tau_{xy}]$
Plane strain	u, v	$[\epsilon_{xx}\ \ \epsilon_{yy}\ \ \gamma_{xy}]$	$[\tau_{xx}\ \ \tau_{yy}\ \ \tau_{xy}]$
Axisymmetric	u, v	$[\epsilon_{xx}\ \ \epsilon_{yy}\ \ \gamma_{xy}\ \ \epsilon_{zz}]$	$[\tau_{xx}\ \ \tau_{yy}\ \ \tau_{xy}\ \ \tau_{zz}]$
Three-dimensional	u, v, w	$[\epsilon_{xx}\ \ \epsilon_{yy}\ \ \epsilon_{zz}\ \ \gamma_{xy}\ \ \gamma_{yz}\ \ \gamma_{zx}]$	$[\tau_{xx}\ \ \tau_{yy}\ \ \tau_{zz}\ \ \tau_{xy}\ \ \tau_{yz}\ \ \tau_{zx}]$
Plate Bending	w	$[\kappa_{xx}\ \ \kappa_{yy}\ \ \kappa_{xy}]$	$[M_{xx}\ \ M_{yy}\ \ M_{xy}]$

Notation: $\epsilon_x = \dfrac{\partial u}{\partial x},\ \epsilon_y = \dfrac{\partial v}{\partial y},\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \ldots, \kappa_{xx} = -\dfrac{\partial^2 w}{\partial x^2},\ \kappa_{yy} = -\dfrac{\partial^2 w}{\partial y^2},\ \kappa_{xy} = 2\dfrac{\partial^2 w}{\partial x\,\partial y}$