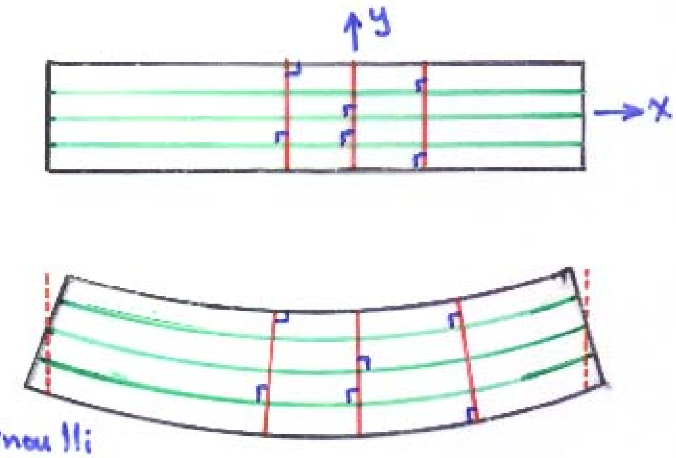
Pure Bending of Prismatic Beams

* The basic 3 ingredients of mechanics are employed here.

* Kinematic Consideration and assumptions:



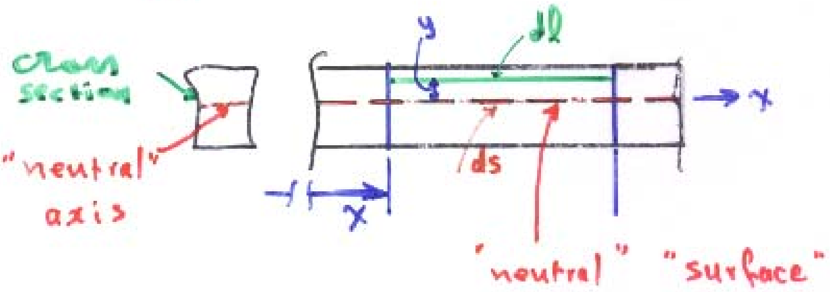
① Transverse cross sections $\perp$ the centroidal line of the beam before bending remain plane and $\perp$ to the centroidal line after deformation (loading) ⇐ Bernoulli hypothesis

② Cross sections are not assumed rigid.
⇒ Lengths of lines differ after deformation.

$$\sigma_y = \sigma_z = 0 \qquad \Rightarrow \qquad \varepsilon_y = \varepsilon_z = -\nu\,\varepsilon_x$$

The assumptions above are geometric, but limited to small deformation.

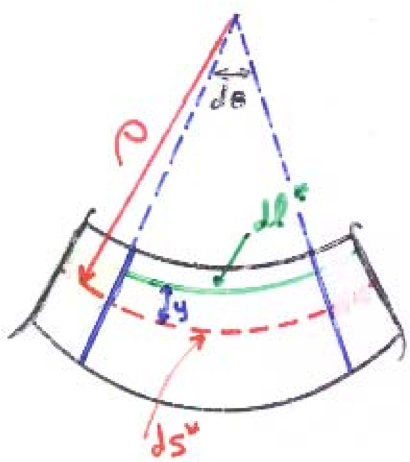
① * Compat.



From the figure:

$$\varepsilon_x = \frac{\delta l}{l} = \frac{dl^* - dl}{dl}$$

Note that
$ds = dl$
$ds^* = ds$
$dl^* \neq dl$



$$\Rightarrow \varepsilon_x = \frac{dl^* - ds}{ds} = \frac{dl^*}{ds} - 1$$

From similar triangles (assuming "small" $d\theta$ ⇒ ds^* and dl^* ≈ straight line)

$$\Rightarrow \frac{dl^*}{\rho - y} = \frac{ds^*}{\rho} = \frac{ds}{\rho}$$

$$\Rightarrow \frac{d\ell^*}{ds} = 1 - \frac{y}{\rho} \quad \Rightarrow \quad \frac{d\ell^*}{ds} - 1 = -\frac{y}{\rho} \quad \Rightarrow \quad \varepsilon_x = -\frac{y}{\rho}$$

ρ = radius of curvature of the deformed centroidal line at section x.

Note that ε_x varies linearly through the depth of the beam.

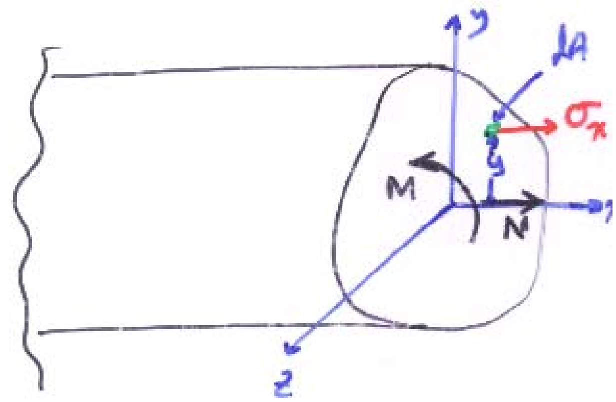
* The assumptions above are geometric and independent of material behavior.

② * Equilibrium

Note that for elastic isotropic materials: $\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$

$\Rightarrow$ Only σ_x is nonzero (pure bending) which is normal to the cross-section.

Thus, for a cross section



$$N = \text{normal force} = \int_{Area} \sigma_x \, dA$$

and

$$M = -\int_A \sigma_x \, y \, dA$$

The equations above are for any material.

③ * Material Behavior

Assume linear elastic behavior $\Rightarrow$

$$\sigma_x = E\,\varepsilon_x = E\left(-\frac{y}{\rho}\right) = -\frac{Ey}{\rho}$$

$$M = -\int_A \sigma_x \, y \, dA = -\int_A \left(-\frac{Ey}{\rho}\right) y \, dA$$

$$= \int_A \frac{E}{\rho} y^2 \, dA$$

Recall from Statics :

$$\int_A y^2 \, dA = I$$

= moment of inertia with respect to the neutral axis (N.A.)

$$\Rightarrow \quad M = \frac{EI}{\rho} \quad \Rightarrow \quad \underline{\underline{\frac{1}{\rho} = \frac{M}{EI}}} \quad \Leftarrow$$ moment-curvature relationship

From above, $\sigma_x = -\frac{Ey}{\rho}$

$$= -Ey\left(\frac{1}{\rho}\right)$$

$$= -Ey\left(\frac{M}{EI}\right) \Rightarrow \boxed{\sigma_x = -\frac{My}{I}}$$ Flexure formula

Note that y is measured from the N.A. ⇒ We need to locate the N.A.

$$N = \int_A \sigma \, dA = -\frac{E}{\rho}\int_A y \, dA = 0$$ ∵ pure bending ⇒

$\frac{E}{\rho} \neq 0$ in general

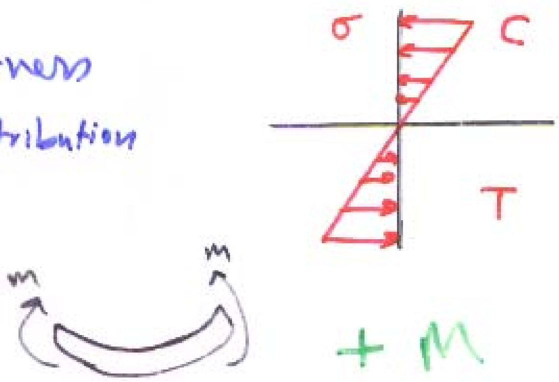
$$\Rightarrow \int_A y \, dA = 0$$

= first moment of area wrt the N.A.

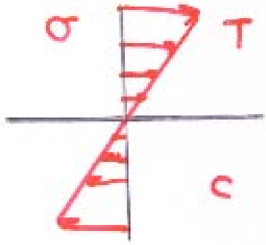
Since it is zero, this means it's the first moment of the area wrt the centroidal axis. ⇒

The neutral axis coincides with the centroidal axis of the cross section for pure elastic bending.

Stress Distribution



+ M



− M