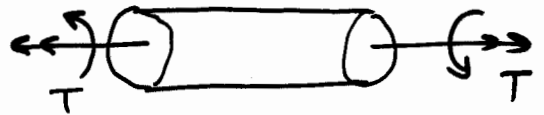


Torsion

There are many torsionally loaded structural elements in life: in airplanes, automobiles, drill equipments, screw drivers,.....



Kinematics of Circular Shafts

* circular shafts: much simpler, more common, very efficient
(You will prove it later! ↓)

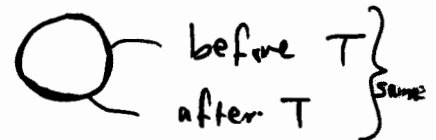
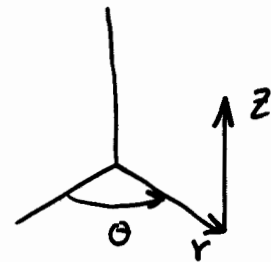
* Cylindrical coordinates system:

* Assumptions:

1) Circular cross sections are rigid
 $\Rightarrow \epsilon_r = \epsilon_\theta = \gamma_{r\theta} = 0$

2) The shaft remains straight and cross-section \perp its geometric axis before deformation remains \perp after deformation. $\Rightarrow 90^\circ$ angles remain $90^\circ \Rightarrow$ no shear
 $\Rightarrow \gamma_{rz} = 0$

3) The distance between cross sections does not change.
 $\Rightarrow \epsilon_z = 0$



The assumptions above are geometric; they do not depend on the material behavior (elastic or inelastic). However, these assumptions are limited to small deformations.

Summary: $\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$

Thus, from Hooke's Law:

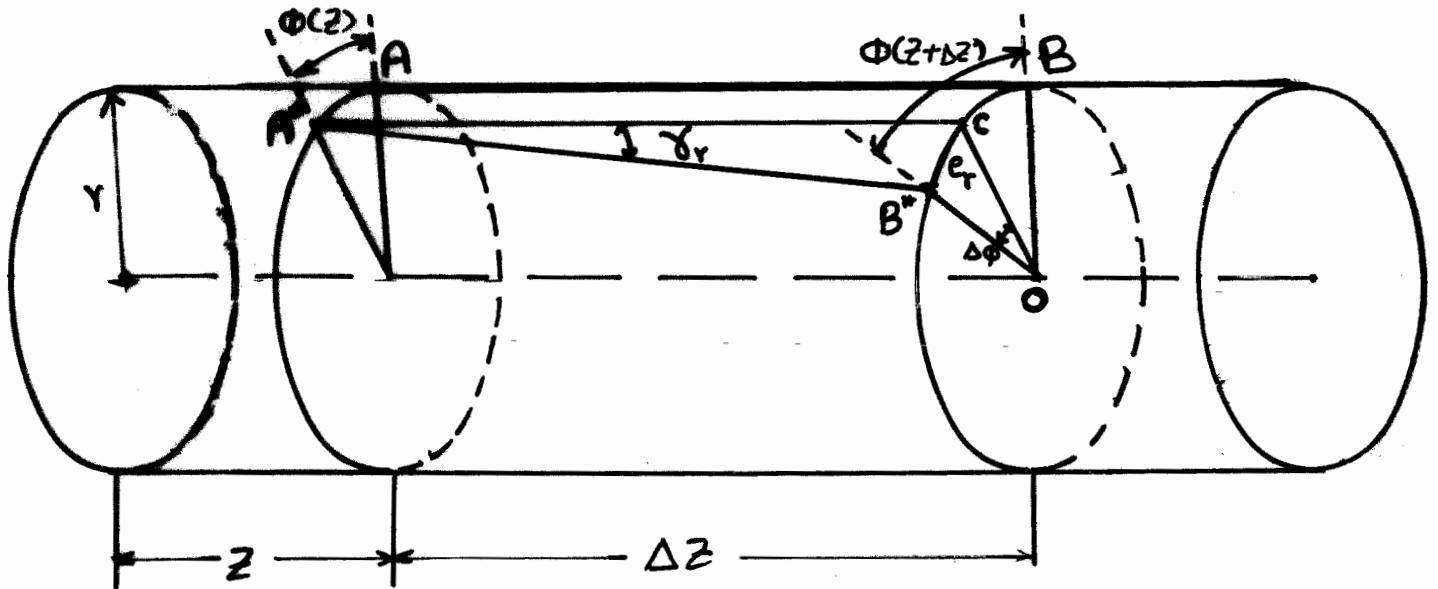
$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$$

Then, we have only

$$\begin{aligned} &\gamma_{\theta z} \\ \Rightarrow &\tau_{\theta z} \end{aligned}$$

Since we have $\gamma_{\theta z}$ and $\tau_{\theta z}$ ($\tau_{z\theta}$) only, we may drop the subscripts. \Rightarrow γ and τ

Kinematics:



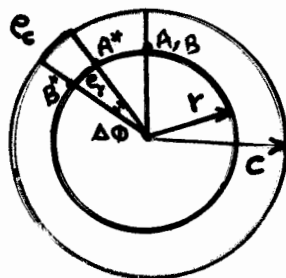
Very small deformation is assumed \Rightarrow

$e_r \approx$ straight line

$e_c \approx$ " "

$\Rightarrow \Delta A^*B^*C$

$\Rightarrow \Delta CB^*O$



Note that e_v is common for the two triangles: A^*B^*C and CB^*O ^{3/16}

$$e_v \approx r \Delta \phi$$

$$= \gamma_v \Delta z$$

$$\Rightarrow r \Delta \phi = \gamma_v \Delta z \quad \Rightarrow \quad \frac{\Delta \phi}{\Delta z} = \frac{\gamma_r}{r}$$

$$\text{Let } \Delta \rightarrow d \quad \Rightarrow \quad \underline{\underline{\frac{d\phi}{dz} = \frac{\gamma_r}{r}}}$$

The equation above expresses the relative rotation of the cross section at $(z+dz)$ wrt the section at z in terms of the shearing strain at a distance r from the center.

Now, we need to find the angle of twist ϕ and the shearing stress/strain τ/γ .

We can not use Statics alone to derive the equations. (TRY!)

Thus, this can be achieved by utilizing:

- ① Equilibrium
- ② Geometric Compatibility
- ③ Material Behavior

The problem is internally SI.

Elastic Twisting of Circular Shafts:

① Equil.:

$$dT = (\tau da) r$$

$$\int dT = \int (\tau da) r$$

$$T = \int_A \tau r da \quad \leftarrow \text{internal}$$

$$((T = \sum M_z)) \quad \leftarrow \text{external}$$



③ Material Behavior:

⇐ Written before ② G. Comp. as it is short & easy. ⇒ go to ②

4/5

$$T = \int \tau r da$$
$$= \int G \gamma r da$$

② Geometric Compatibility:

$$T = \int G \gamma r da$$
$$= \int G \frac{\gamma}{r} r^2 da$$
$$= \int G \frac{d\phi}{dz} r^2 da$$
$$= G \frac{d\phi}{dz} \int r^2 da$$

$\frac{d\phi}{dz}$ is the unit angle of twist

Recall from Statics that $\int r^2 da = J$

$J =$ polar moment of inertia wrt the z -axis

Thus:

$$\frac{d\phi}{dz} = \frac{T}{JG}$$

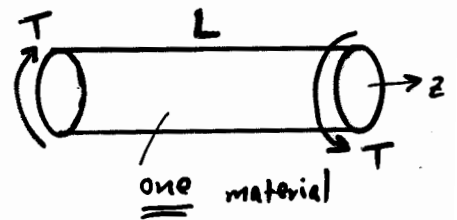
$$J = \frac{\pi}{2} r^4$$

for solid section

$$J = \frac{\pi}{2} (r_{out}^4 - r_{in}^4)$$

for hollow section

For uniform shafts (constant T, J, G) as shown:



$$d\phi = \frac{T}{JG} dz$$
$$\int_{z_i}^{z_{i+1}} d\phi = \int_{z_i}^{z_{i+1}} \frac{T}{JG} dz$$
$$\phi_{i+1} - \phi_i = \frac{T}{JG} \int_{z_i}^{z_{i+1}} dz$$

Thus, for a uniform shaft:

$$\phi = \frac{TL}{JG}$$

⇐ total angle of twist!

For nonuniform shafts, $\frac{T}{JG}$ may contain discontinuities caused by abrupt changes in the cross section, the applied torques and/or in the material.

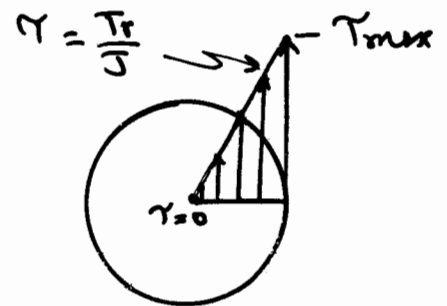
The three methods of analysis [① direct integration, ② discrete element, and ③ superposition] discussed earlier in axially-loaded members problems can be used here.

$$\underline{\underline{\Phi_{total} = \sum \Phi_i = - \sum \frac{TL}{JG}}}$$

Shearing Stress (τ):

$$\begin{aligned} \tau &= G \gamma \\ &= G \frac{\gamma}{r} r \\ &= G \frac{d\theta}{dz} r \\ &= G \left(\frac{T}{JG} \right) r \end{aligned}$$

$$\Rightarrow \boxed{\tau = \frac{Tr}{J}}$$



\leftarrow τ is any radius in the shaft

$$\boxed{\tau_{max} = \frac{T r_{max}}{J}}$$

\Leftarrow τ_{max} is usually the outside radius r_{out}
 $\tau = 0$ @ $r = 0$

\Uparrow
 This same formula gives τ_{max} @ r_{max} in the section.

τ_{max} may also be at other sections with T_{max} and/or J_{min} .