Moments of Inertia « I »

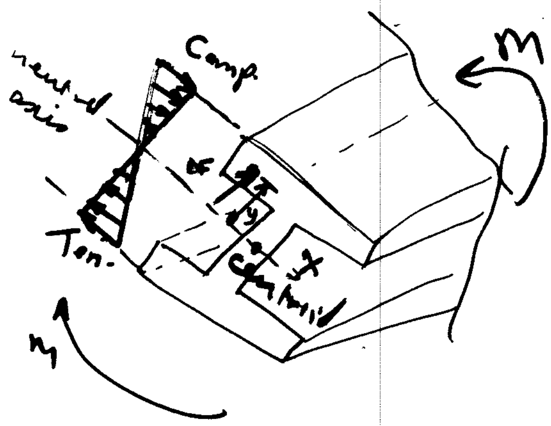
(Second moment of the Area)

In the previous chapter « centroid », $\Delta W \propto \Delta A$

$\Rightarrow$ $\int dA$ and $\int y\,dA$ were found and calculated.

Sometimes, ΔF magnitude depends on ΔA as well as the distance from ΔA to some given axis.

$\Delta F = ky\,\Delta A$



The resultant R of the forces ΔF over the entire cross-section is

$$R = \int \Delta F = \int ky\,dA$$
$$= k \int_A y\,dA$$

$\int y\,dA = \bar{y} A$ = first moment of the cross-sectional area about the x-axis

$= 0$ (since the centroid is located on the x-axis.)

The moment about the x-axis:

$$\Delta M_x = (\Delta F)\,y = (ky\,\Delta A)\,y = ky^2\,\Delta A$$

$$M_x = \int \Delta M_x = \int_A ky^2\,dA = k \boxed{\int_A y^2\,dA}$$

$\int y^2\,dA$ = moment of inertia (or second moment of the cross-sectional area) w.r.t. to the x-axis

$= I_x$ ← always ⊕

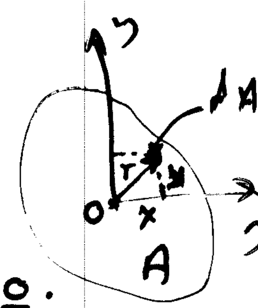
$$I_x = \int y^2 \, dA$$
$$I_y = \int x^2 \, dA$$

$\Leftarrow$ units $= l^4$ (e.g. m^4, in^4, ...etc.)

I_x and I_y can be determined by integration. This is pure math, as you can see the handout.

Polar moment of Inertia J_o :

$$J_o = \int r^2 \, dA$$



J_o is the polar moment of inertia of A wrt $\underline{O}$.

$$J_o = \int r^2 \, dA$$
$$= \int (x^2 + y^2) \, dA = \int y^2 \, dA + \int x^2 \, dA$$

$$\Rightarrow \quad J_o = I_x + I_y$$

Radius of gyration k :

$$k_x = \sqrt{I_x / A}$$

$$k_y = \sqrt{I_y / A}$$

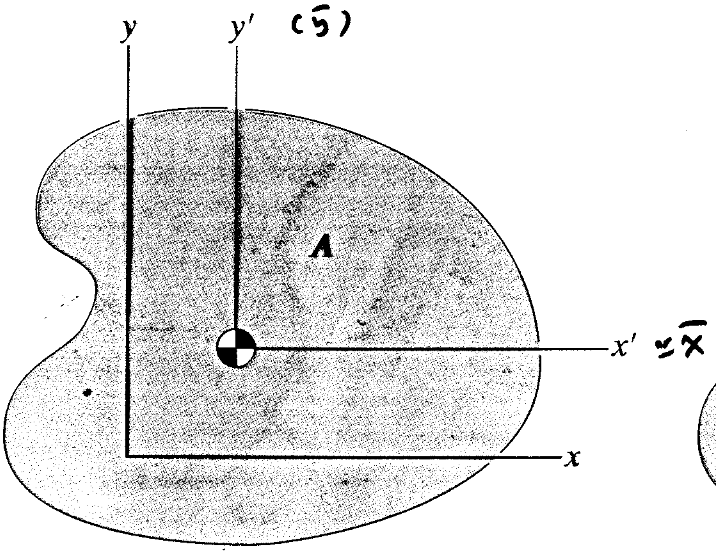
$$k_o = \sqrt{J_o / A}$$

$$\Rightarrow \quad k_o^2 = k_x^2 + k_y^2$$

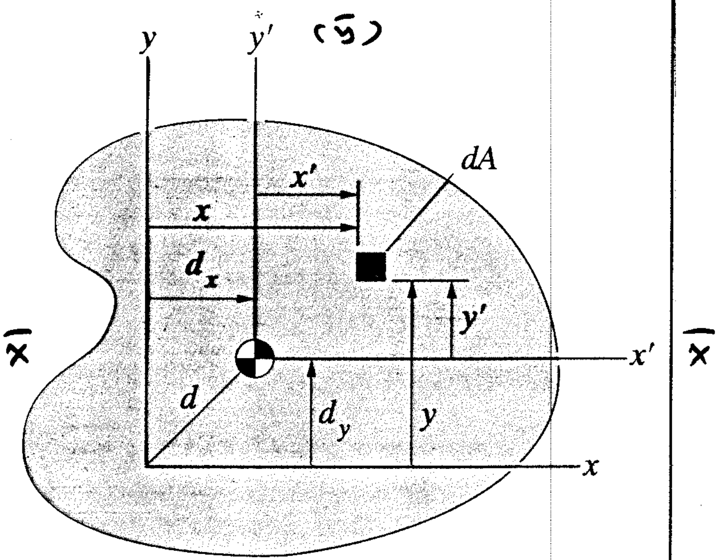
Parallel-axis Theorem

$I_x = \int y^2 \, dA$

x' or $\bar{x}$ is the centroidal axis.



(a)



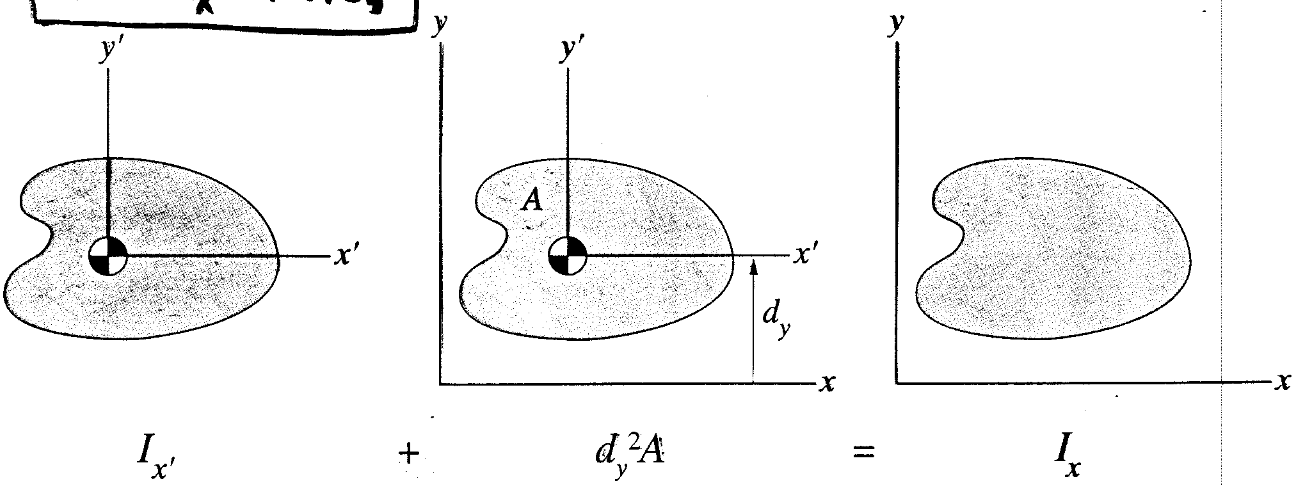
(b)

$$I_x = \int y^2 \, dA = \int (y' + d_y)^2 \, dA = \int [(y')^2 + 2y'd_y + (d_y)^2] \, dA$$

$$= \int y'^2 \, dA + \int 2 d_y y' \, dA + \int d_y^2 \, dA$$

$$= \int y'^2 \, dA + 2 d_y \underbrace{\int y' \, dA}_{0} + d_y^2 \int dA$$

$$\boxed{I_x = \bar{I}_{x'} + A d_y^2}$$



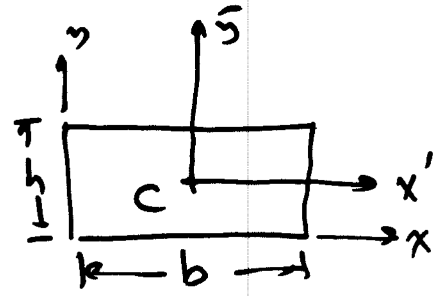
Similarly:

$$\boxed{I_y = \bar{I}_{y'} + A d_x^2} \quad ; \quad \boxed{J_0 = \bar{J}_0 + A d^2}$$

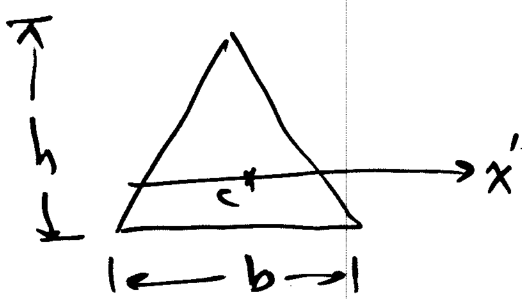
$$\bar{I}_{x'} = \frac{1}{12} b h^3$$

$$\bar{I}_{y'} = \frac{1}{12} h b^3$$

$$\left(\left(\begin{aligned} I_x &= \frac{1}{3} b h^3 \\ I_y &= \frac{1}{3} h b^3 \end{aligned} \right)\right)$$

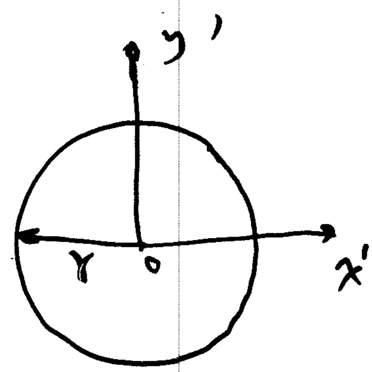


$$\bar{I}_{x'} = \frac{1}{36} b h^3$$

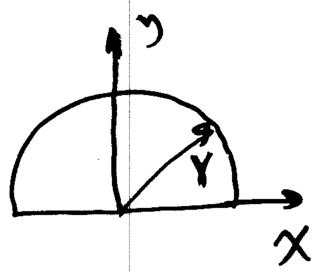


$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{\pi}{4} r^4$$

$$J_o = \frac{\pi}{2} r^4$$



$$I_x = I_y = \frac{\pi}{8} r^4$$



not centroidal axis ⇒ ?!!!

Moments of Inertia by the Composite Area Method

The total area A can be divided into areas A_1, A_2, A_3, ..., A_n, then

$$\int y^2\,dA = \int y^2\,dA_1 + \int y^2\,dA_2 + \dots \int y^2\,dA_n$$

$$= I_{x_{①}} + I_{x_{②}} + \dots + I_{x_{ⓝ}}$$

$$= \left[\left(\bar{I}_{x'_{①}} + A_{①} d_{①}^2\right) + \dots + \left(\bar{I}_{x'_{ⓝ}} + A_{ⓝ} d_{ⓝ}^2\right)\right]$$

$$I_x = \sum_{i=1}^{n} \left[\bar{I}_{x'_i} + (A d^2)_i\right]$$

also

$$\bar{I}_{x'} = \sum \left[\bar{I}_{x'_i} + (A d^2)_i\right]$$

What is the difference ?!! in d ⇓ See the definition in the handout !

Similarly,

$$I_y \text{ or } \bar{I}_{y'} = \sum \left[\bar{I}_{y'_i} + (A d^2)_i\right]$$

See the table on the inside back cover of your textbook for the values of $\left\{\begin{matrix} \bar{I}_{x'} \text{ and } \bar{I}_{y'} \\ \text{or } I_x \text{ and } I_y \end{matrix}\right\}$ for common area shapes.

* Very Important:

Do NOT get confused between $x'-y'$ and $x-y$!!!!

See, read, and understand the handout !!!