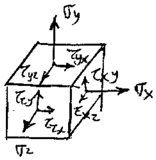
Chapter 7.

Stress Transformation & Deflections

Two - Dimensional Transformation of stress

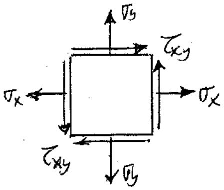


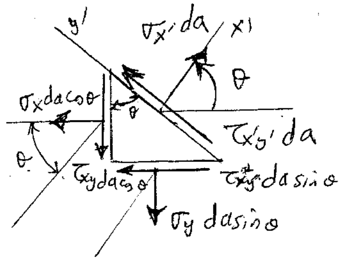
$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{array}{l} \leftarrow \text{x-plane} \\ \leftarrow \text{y-plane} \\ \leftarrow \text{z-plane} \end{array}$$

$\sigma_{all} = 25\,kPa$

$\tau_{all} = 5\,kPa$

Is it safe





$\Sigma F_{x'} = 0$: $\sigma_{x'} da - \tau_{xy} da \sin\theta \cos\theta - \tau_{xy} da \cos\theta \sin\theta$
$- \sigma_x da \cos\theta \cos\theta - \sigma_y da \sin\theta \sin\theta = 0$

$$\sigma_{x'} - \sigma_x \cos^2\theta - \sigma_y \sin^2\theta - 2\tau_{xy} \sin\theta \cos\theta = 0$$

$$\Rightarrow \quad \sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta \quad — (1)$$

$\Sigma F_{y'} = 0$; $\quad \tau_{x'y'} = -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad — (2)$

using the following into (1) & (2) $\Rightarrow$

$$\left.\begin{array}{l} \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \\ \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \\ \sin\theta \cos\theta = \frac{1}{2}\sin 2\theta \\ \cos^2\theta - \sin^2\theta = \cos 2\theta \end{array}\right\} \Rightarrow$$

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta} \quad — (A)$$

$$\boxed{\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cdot \cos 2\theta} \quad — (B)$$

to find $\sigma_{y'}$, use (A) with $\theta = \theta + \pi/2 \Rightarrow$

$$\boxed{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta - \tau_{xy} \sin 2\theta} \quad — (C)$$

Also $\quad \sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$

Principal Values and Principal Directions

To find the maximum and minimum values of the normal stress $\sigma_x' = \sigma_x'(\underbrace{\sigma_x, \sigma_y, \tau_{xy}}_{\text{constants}}, \theta)$, one needs to set $\frac{d\sigma_x'}{d\theta} = 0$ and find θ_p then substitute to find σ_{max} and σ_{min}.

$$\frac{d\sigma_x'}{d\theta} = 0 \Rightarrow \boxed{\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}},$$

which upon substitution into (A), gives

$$\boxed{\left\{\begin{matrix}\sigma_{max} \\ \sigma_{min}\end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

which from (B) gives $\tau = 0$

$\therefore$ Planes of principal normal stress have no shear stress

Similarly to find maximum shear stress set

$$\frac{d\tau_{x'y'}}{d\theta} = 0 \Rightarrow \boxed{\tan 2\theta_s = \frac{-\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}}}$$

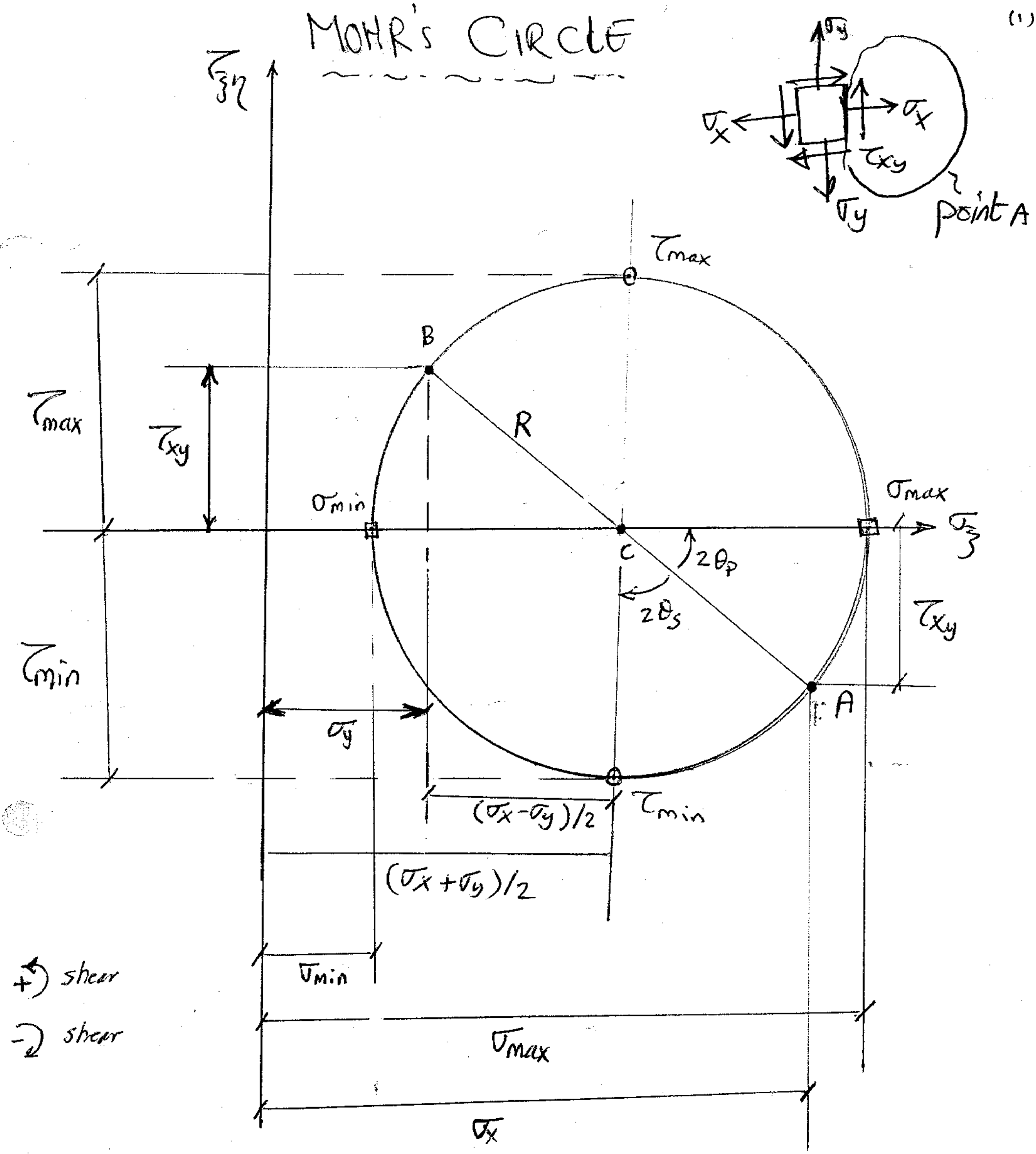
which upon substitution into (B) gives

$$\boxed{\left\{\begin{matrix}\tau_{max} \\ \tau_{min}\end{matrix}\right\} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

which upon substitution into (A) gives

$\sigma = \frac{\sigma_x + \sigma_y}{2}$, the corresponding normal stress on planes of principal shear stress.

MOHR'S CIRCLE



$$C = \left(\frac{\sigma_x + \sigma_y}{2},\ 0 \right)$$

$$A = (\sigma_x, -\tau_{xy})$$

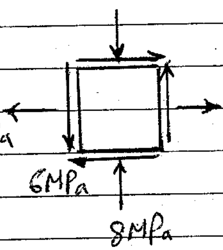
$$R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2} - \sigma_x \right)^2 + \tau_{xy}^2} = \text{distance between C and A}$$

Given the state of stress at a material point as shown below, find the following in two ways: using equations and Mohr's Circle.

[1] Find the principal normal stress (magnitude + direction) show them on properly oriented element with respect to x-y axes.

[2] Find the principal shear stress (magnitude + direction) show them on properly oriented element. DO NOT FORGET the normal stresses that are acting with shear.

[3] Find the stress state (normal and shear stresses) at plane making an angle $+30°$.

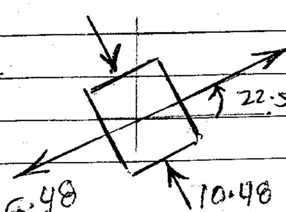


$\sigma_x = 4\,MPa$ $\sigma_y = -8\,MPa$ $\tau_{xy} = 6\,MPa$

Equation Method:

[1] $\sigma_{\substack{max\\min}} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \Rightarrow \sigma_{\substack{max\\min}} = \begin{cases} 6.48 \\ -10.48 \end{cases}$

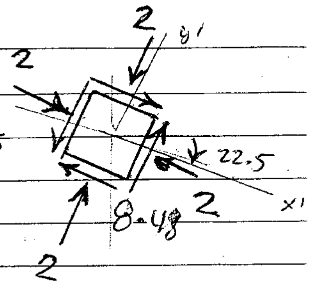
$\tan 2\theta_p = \dfrac{\tau_{xy}}{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)} \Rightarrow \theta_p = 22.5$



[2] $\tau_{\substack{max \\ Min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm 8.48$

$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} \Rightarrow \theta_s = -22.5$

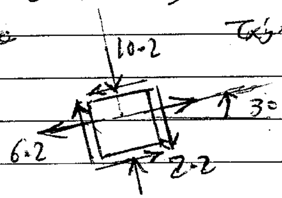
$\sigma_{avg.} = \frac{\sigma_x + \sigma_y}{2} = -2$



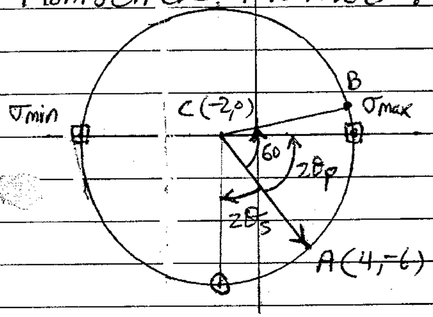
[3] $\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \cdot 2\theta \Rightarrow \sigma_{x'} = 6.2$

$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta$ $\quad \tau_{x'y'} = -2.2$

$\sigma_{y'} = \sigma_x + \sigma_y - \sigma_{x'} = -10.2$



Mohr's circle Method :



R = distance between C and A

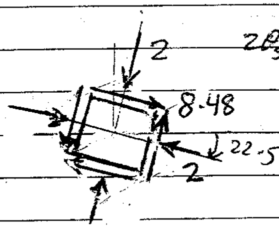
$= \sqrt{(-2-4)^2 + 6^2} = 8.48$

[1] $\sigma_{max} = C + R = 6.48$

$\sigma_{min} = C - R = -10.48$

$\sin 2\theta_p = \frac{6}{8.48} \Rightarrow \theta_p = 22.5$

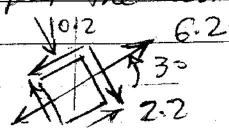
[2] $\tau_{\substack{max \\ Min}} = \pm R = \pm 8.48$ $\qquad 2\theta_s = 90 - 2\theta_p \Rightarrow \theta_s = 22.5$



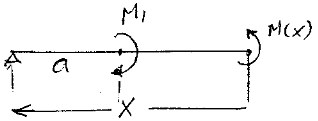
[3] In order to find the stress at a plane $+30°$, you need to go from A counterclockwise $60°$, which gives point B. Then stresses at the plane nothing but the coordinate of B:

$\sigma = R\cos 15 - 2 = 6.2$

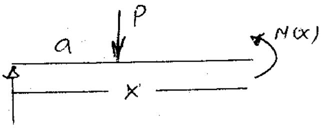
$\tau = R \sin 15 = +2.2$



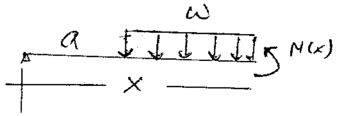
Singularity Function:



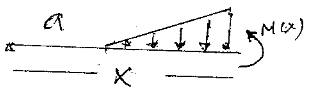
$$M(x) = +M_1 \langle x-a\rangle^0$$



$$M(x) = -P\langle x-a\rangle^1$$



$$M(x) = -\frac{\omega}{2}\langle x-a\rangle^2$$



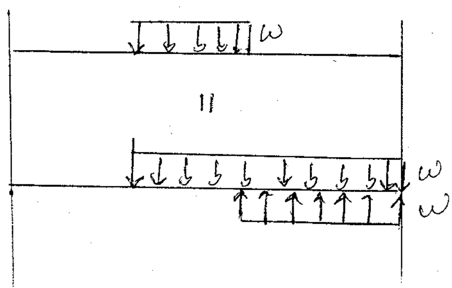
$$M(x) = -\frac{k}{6}\langle x-a\rangle^3$$

Macaulay brackets : $\langle +\alpha\rangle = +\alpha$

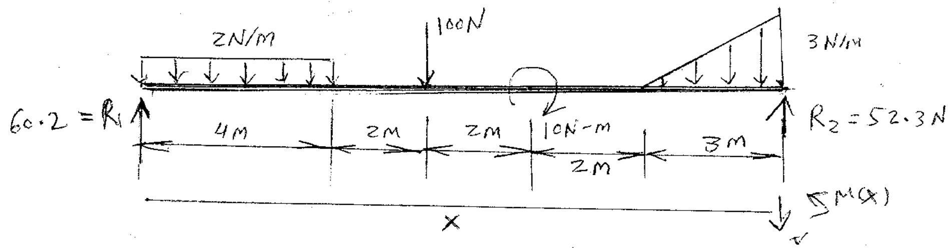
$\langle -\alpha\rangle = 0$

Remember distributed loads can start anywhere but should continue to the end of the beam; otherwise it should be forced to go to end.

Ex.



Example : Find the slope of the beam at support A



$$EI\,v''(x) = 60.2\langle x-0\rangle - \frac{2}{2}\langle x-0\rangle^2 + \frac{2}{2}\langle x-4\rangle^2 - 100\langle x-6\rangle + 10\langle x-8\rangle^0 - \frac{3/3}{6}\langle x-10\rangle^3$$

$$EI\,v'(x) = \frac{60.2}{2}\langle x-0\rangle^2 - \frac{1}{3}\langle x-0\rangle^3 + \frac{1}{3}\langle x-4\rangle^3 - 50\langle x-6\rangle^2 + 10\langle x-8\rangle - \frac{1}{24}\langle x-10\rangle^4 + C_1$$

$$EI\,v(x) = \frac{60.2}{6}\langle x-0\rangle^3 - \frac{1}{12}\langle x-0\rangle^4 + \frac{1}{12}\langle x-4\rangle^4 - \frac{50}{3}\langle x-6\rangle^3 + \frac{10}{2}\langle x-8\rangle^2 - \frac{1}{24\times5}\langle x-10\rangle^5 + C_1\langle x-0\rangle + C_2$$

B.C.

$v(0)=0 \Rightarrow C_2 = 0$

$v(13)=0 \Rightarrow C_1 = -1124$

$\therefore$

$$\boxed{v(x) = \frac{1}{EI}\left\{\frac{60.2}{6}\langle x-0\rangle^3 - \frac{1}{12}\langle x-0\rangle^4 + \frac{1}{12}\langle x-4\rangle^4 - \frac{50}{3}\langle x-6\rangle^3 + 5\langle x-8\rangle^2 - \frac{1}{120}\langle x-10\rangle^5 - 1124\langle x-0\rangle\right.}$$

$v'(x=0)$ = Slope at Support A

$= C_1/EI$

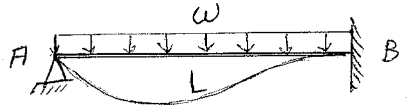
$= -1124/EI$

θ

$+\theta$

Using Singularity Functions to Determine Reactions of Statically Indeterminate Problems :

Example ① : Find reaction at support A



$$EI\,v''(x) = M(x)$$

$$EI\,v''(x) = A_y \langle x-0\rangle - \frac{w}{2}\langle x-0\rangle^2$$

$$EI\,v'(x) = \frac{A_y}{2}\langle x-0\rangle^2 - \frac{w}{6}\langle x-0\rangle^3 + C_1$$

$$EI\,v(x) = (A_y/6)\langle x-0\rangle^3 - \frac{w}{24}\langle x-0\rangle^4 + C_1\langle x-0\rangle + C_2$$

unknowns : $\{C_1, C_2, A_y\}$

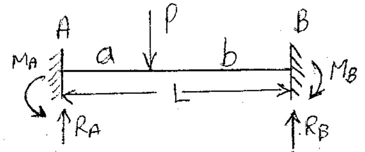
Boundary Conditions :

$$\begin{cases} v(0)=0 & \Rightarrow & C_2 = 0 & \text{— (1)} \\ v(L)=0 & \Rightarrow & A_y L^3/6 - \frac{wL^4}{24} + C_1 L = 0 & \text{— (2)} \\ v'(L)=0 & \Rightarrow & \frac{A_y}{2}L^2 - \frac{wL^3}{6} + C_1 = 0 & \text{— (3)} \end{cases}$$

solving ① – ③ $\Rightarrow$

$$C_1 = -wL^3/48 \qquad C_2 = 0 \qquad \underline{A_y = \tfrac{3}{8}wL}$$

Example ②



Find reactions at A & B

$$EI\,v''(x) = R_A\langle x-0\rangle - M_A\langle x-0\rangle^0 - P\langle x-a\rangle$$

$$EI\,v'(x) = \frac{R_A}{2}\langle x-0\rangle^2 - M_A\langle x-0\rangle - \frac{P}{2}\langle x-a\rangle^2 + C_1$$

$$EI\,v(x) = (R_A/6)\langle x-0\rangle^3 - (M_A/2)\langle x-0\rangle^2 - (P/6)\langle x-a\rangle^3 + C_1\langle x-0\rangle + C_2$$

unknowns : $\{C_1, C_2, R_A, M_A\}$

Boundary conditions : $\{ v(0)=0 \,;\; v'(0)=0 \,;\; v(L)=0 \;\; v'(L)=0 \}$

$v(0)=0$: $C_2 = 0$

$v'(0)=0$; $C_1 = 0$

$$\left.\begin{aligned} v(L)=0 &: \frac{R_A}{6}L^3 - \frac{M_A}{2}L^2 - \frac{P}{6}b^3 = 0 \\ v'(L)=0 &; \frac{R_A}{2}L^2 - M_A L - \frac{P}{2}b^2 = 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} R_A &= \frac{Pb^2}{L^3}(3L-2b) \\ M_A &= Pab^2/L^2 \end{aligned}$$

one can use equilibrium equations to find the remaining unknowns $(R_B \,\&\, M_B)$;